Vectors

Equation of a line

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{d} \Leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \Leftrightarrow \frac{x-a_1}{d_1} = \frac{y-a_2}{d_2} = \frac{z-a_3}{d_3}$$

The first equation is called the parametric form of a line, and the one in quotients is called the Cartesian form of a line (in 3D). Note that the Cartesian form has *no* coefficient before the variables and that the fixed point is *subtracted*. i.e. $\frac{5x+3}{4}$ should be converted to $\frac{x-\left(-\frac{3}{5}\right)}{\frac{4}{5}}$ giving $a_1 = -\frac{3}{5}, d_1 = \frac{4}{5}$; not $a_1 = 3, d_1 = 4$.

The scalar and vector product (or the dot and cross product)

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Two vectors that are perpendicular have a zero scalar product.

The projection of one vector onto another is given by $\underset{\sim}{a} \cdot \frac{\underset{\sim}{b}}{|\underset{\sim}{b}|}$.

$$\underset{\sim}{a} \times \underset{\sim}{b} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}, \ |\underset{\sim}{a} \times \underset{\sim}{b}| = |\underset{\sim}{a}||\underset{\sim}{b}|\sin\theta$$

The vector product of $\underset{\sim}{a}$ and $\underset{\sim}{b}$ generates a vector that is perpendicular to the plane containing $\underset{\sim}{a}$ and $\underset{\sim}{b}$, i.e. a normal vector.

Equation of a plane

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{p} + \mu \underset{\sim}{q} \Leftrightarrow (\underset{\sim}{r} - \underset{\sim}{a}) \cdot \underset{\sim}{n} = 0 \Leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = -d \Leftrightarrow n_1 x + n_2 y + n_3 z + d = 0$$

The first equation is called the parametric form of a plane, the second equation is called the normal vector form of a plane, and the last equation is called the Cartesian form of a plane (in 3D). The normal vector is easily generated from the vector product.

IB Notes by Jonathan Hsu, 2nd Edition, 2000

Perpendicular distance from a point to a line

The perpendicular distance from (x_1, y_1, z_1) to $\underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{d}$ is $\left| \dfrac{\left(\underset{\sim}{p} - \underset{\sim}{a}\right) \cdot \underset{\sim}{d}}{\left|\underset{\sim}{d}\right|} \right|$

Perpendicular distance between two lines

$L_1: \quad \dfrac{x-a_1}{d_1} = \dfrac{y-a_2}{d_2} = \dfrac{z-a_3}{d_3}$ $\qquad L_2: \quad \dfrac{x-b_1}{e_1} = \dfrac{y-b_2}{e_2} = \dfrac{z-b_3}{e_3}$

(1) Find a vector perpendicular to both lines using the vector product,

$$\underset{\sim}{b} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ d_1 & d_2 & d_3 \\ e_1 & e_2 & e_3 \end{vmatrix}$$

(2) Find a vector connecting any two points on the two lines,

$$L_1 : (x_1, y_1, z_1),\ L_2 : (x_2, y_2, z_2) \Rightarrow \underset{\sim}{a} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} - \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

(3) The perpendicular distance between the two lines is the projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$, $\left| \dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\left|\underset{\sim}{b}\right|} \right|$

Perpendicular distance from a point to a plane

The perpendicular distance from (x_1, y_1, z_1) to the plane $ax + by + cz + d = 0$ is $\left| \dfrac{ax_1 + by_1 + cz_1 - d}{\sqrt{a^2 + b^2 + c^2}} \right|$. (If plane is given in scalar product form, change it into Cartesian form then use this formula.)

Intersection of a line and a plane

$\dfrac{x-x_1}{e_1} = \dfrac{y-y_1}{e_2} = \dfrac{z-z_1}{e_3}$ $\qquad ax + by + cz = d$

(1) Introduce a parameter and express the points on the line as $x = e_1\lambda + x_1$, $y = e_2\lambda + y_1$ and $z = e_3\lambda + z_1$
(2) Substitute these expressions into the equation of the plane and solve for λ
(3) The point of intersection is found by substituting λ back into the parametric equations in (1)

Intersection of two lines

$L:\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$ $\qquad$ $M:\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \mu \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$

(1) Solve the simultaneous equations $a_1 + \lambda d_1 = b_1 + \mu e_1$ and $a_2 + \lambda d_2 = b_2 + \mu e_2$ for λ, μ.
(2) Substitute this pair of values (λ, μ) into the two equations for the *z*-coordinate
(3) If the *z*-coordinate for the two lines equal given λ, μ, then the lines intersect at a point; otherwise they do not intersect.

Note that if the direction vector of one line is a scalar multiple of the other, $\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = n \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, n \in \mathbb{R}$, then the two lines are parallel. If $\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \neq n \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$ and the two lines do not intersect they are said to be *skew*.

Intersection of two planes

$ax + by + cz = d$ $\qquad$ $ex + fy + gz = h$

(1) First eliminate *z* from the two equations, expressing *x* in terms of *y* only
(2) Then eliminate *y* from the two equations, expressing *x* in terms of *z* only
(3) Combining the two expressions give an equation of a line $x = \alpha(y) = \beta(z)$

Intersection of three planes

$a_1x + b_1y + c_1z = d_1$ $\qquad$ $a_2x + b_2y + c_2z = d_2$ $\qquad$ $a_3x + b_3y + c_3z = d_3$

$$M = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \Rightarrow M \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

(1) The three planes will intersect at a point if the matrix *M* is non-singular, i.e. $\det M \neq 0$, in which case the point of intersection can be found by row reduction or the inverse matrix
(2) If the matrix *M* is singular, then either there is no solution (two inconsistent equations when one variable is eliminated in two different ways) or there are infinitely many solutions (two equivalent equations when one variable is eliminated in two different ways)