Proving Derivatives from First Principles

A. Definition

The derivative of a function *f* at any given value *x* in the domain is defined as:

$$\frac{d}{dx}f(x)=f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

provided this limit exists.

B. Basic Derivatives

Derivative of a constant

If $f(x)=c$, where *c* is a constant independent of *x*, then $f'(x)=0$.

Proof

$$\begin{aligned}f'(x)&=\lim_{h\to 0}\frac{c-c}{h}\\&=\lim_{h\to 0}0=0\end{aligned}$$

The power rule for positive integer powers

The case for the power being 1 is very simple. It is consistent with the idea of the gradient of the line $y=x$ being equal to 1.

If $f(x)=x$, then $f'(x)=1$.

Proof

$$\begin{aligned}f'(x)&=\lim_{h\to 0}\frac{(x+h)-x}{h}\\&=\lim_{h\to 0}\frac{h}{h}\\&=\lim_{h\to 0}1=1\end{aligned}$$

If $f(x)=x^n$, then $f'(x)=nx^{n-1}$.

The case for higher powers requires more work. Those who have done a course on calculus might be tempted to prove this result using the product rule repeatedly. Although it is not considered to be a proper "proof from first principles", there is nothing stopping you to try it as an exercise on mathematical induction.

Proof (Method One – Binomial Theorem)
Recall the Binomial Theorem for positive integers:

$$(a+b)^n = \sum_{i=0}^{i=n} \binom{n}{i} a^{n-i} b^i$$

Applying this to the derivative function gives:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h\to 0} \frac{\left(x^n + nx^{n-1}h + \tfrac{1}{2}n(n-1)x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n\right) - x^n}{h} \\
&= \lim_{h\to 0} \frac{nx^{n-1}h + \tfrac{1}{2}n(n-1)x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n}{h} \\
&= \lim_{h\to 0}\left(nx^{n-1} + \tfrac{1}{2}n(n-1)x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}\right) \\
&= \lim_{h\to 0} nx^{n-1} + \lim_{h\to 0} \tfrac{1}{2}n(n-1)x^{n-2}h + \cdots + \lim_{h\to 0} nxh^{n-2} + \lim_{h\to 0} h^{n-1} \\
&= nx^{n-1} + 0 + \cdots + 0 + 0 \\
&= nx^{n-1}
\end{aligned}$$

Proof (Method Two – Change of Variable)
Let $a = x + h$.
Then $h = a - x$ and $h \to 0 \Rightarrow a - x \to 0 \Rightarrow a \to x$.
So the derivative can be written in the form:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{a\to x} \frac{f(a) - f(x)}{a - x}
\end{aligned}$$

Now $a^n - x^n = (a-x)\left(a^{n-1} + a^{n-2}x + \cdots + ax^{n-2} + x^{n-1}\right)$

So the derivative becomes:

$$\begin{aligned}
f'(x) &= \lim_{a\to x} \frac{f(a) - f(x)}{a - x} \\
&= \lim_{a\to x} \frac{a^n - x^n}{a - x} \\
&= \lim_{a\to x}\left(a^{n-1} + a^{n-2}x + \cdots + ax^{n-2} + x^{n-1}\right) \\
&= \lim_{a\to x} a^{n-1} + \lim_{a\to x} a^{n-2}x + \cdots + \lim_{a\to x} ax^{n-2} + \lim_{a\to x} x^{n-1} \\
&= x^{n-1} + x^{n-2}x + \cdots + xx^{n-2} + x^{n-1} \\
&= nx^{n-1}
\end{aligned}$$

Note that these proofs make use of various properties of limits. If you are not familiar with these properties (or how they are derived) please refer to "Laws of Limits".

C. Linear Combinations of Derivatives

The constant multiplier rule

If c is a constant and f is a differentiable function, then $\frac{d}{dx}\left[cf(x)\right] = cf'(x)$.

Proof

Let $g(x) = cf(x)$.

$$\begin{aligned} g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\ &= \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h} \\ &= \lim_{h\to 0} c\left[\frac{f(x+h)-f(x)}{h}\right] \\ &= c\lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right] \\ &= cf'(x) \end{aligned}$$

The sum rule

If f and g are both differentiable, then $\frac{d}{dx}\left[f(x)+g(x)\right] = f'(x)+g'(x)$.

Proof

Let $F(x) = f(x)+g(x)$.

$$\begin{aligned} F'(x) &= \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} \\ &= \lim_{h\to 0}\frac{\left[f(x+h)+g(x+h)\right]-\left[f(x)+g(x)\right]}{h} \\ &= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right] \\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\ &= f'(x)+g'(x) \end{aligned}$$

The difference rule

If f and g are both differentiable, then $\frac{d}{dx}\left[f(x)-g(x)\right]=f'(x)-g'(x)$.

Proof
Write $f(x)-g(x)=f(x)+(-1)g(x)$.
Then by the sum rule:

$$\begin{aligned}\frac{d}{dx}\left[f(x)-g(x)\right]&=\frac{d}{dx}\left[f(x)+(-1)g(x)\right]\\&=f'(x)+\frac{d}{dx}\left[(-1)g(x)\right]\end{aligned}$$

But by the constant multiplier rule:

$$\begin{aligned}\frac{d}{dx}\left[(-1)g(x)\right]&=-\frac{d}{dx}g(x)\\&=-g'(x)\end{aligned}$$

So combining the two results give:

$$\begin{aligned}\frac{d}{dx}\left[f(x)-g(x)\right]&=f'(x)+\frac{d}{dx}\left[(-1)g(x)\right]\\&=f'(x)+\left(-g'(x)\right)\\&=f'(x)-g'(x)\end{aligned}$$

D. The Product Rule and the Quotient Rule

The Product Rule

If *f* and *g* are both differentiable and their product is defined for all values of *x* in the domain, then $\frac{d}{dx}\left[f(x)g(x)\right]=f(x)g'(x)+g(x)f'(x)$.

Proof
Let $F(x)=f(x)g(x)$.

$$\begin{aligned}F'(x)&=\lim_{h\to0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}\\&=\lim_{h\to0}\frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h}\\&=\lim_{h\to0}\left[f(x+h)\frac{g(x+h)-g(x)}{h}+g(x)\frac{f(x+h)-f(x)}{h}\right]\\&=\lim_{h\to0}f(x+h)\frac{g(x+h)-g(x)}{h}+\lim_{h\to0}g(x)\frac{f(x+h)-f(x)}{h}\end{aligned}$$

Since $g(x)$ is a constant with respect to the variable *h*, $\lim_{h\to0}g(x)=g(x)$.

Since *f* is differentiable, it follows that it is also continuous hence by definition of a continuous function $\lim_{h\to 0} f(x+h) = f(x)$.

Now substituting these two limits into the derivative gives:

$$F'(x) = \lim_{h\to 0} f(x+h) \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} + \lim_{h\to 0} g(x) \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$
$$= f(x)g'(x) + g(x)f'(x)$$

The Quotient Rule

If *f* and *g* are both differentiable, and that the function *g* does not equal to zero for all values of x in the domain, then $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x)-f(x)g'(x)}{\left[g(x)\right]^2}$.

Proof

Let $F(x) = f(x)/g(x)$.

$$F'(x) = \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$
$$= \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{g(x)g(x+h)h}$$
$$= \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{g(x)g(x+h)h}$$
$$= \lim_{h\to 0} \frac{g(x)\frac{f(x+h)-f(x)}{h} - f(x)\frac{g(x+h)-g(x)}{h}}{g(x)g(x+h)}$$

Since $f(x), g(x)$ are constants with respect to the variable *h*, the limits equal the functions; $\lim_{h\to 0} f(x) = f(x), \lim_{h\to 0} g(x) = g(x)$.

Since *g* is differentiable, it follows that it is also continuous hence by definition of a continuous function $\lim_{h\to 0} g(x+h) = g(x)$.

Substituting these limits into the derivative gives:

$$F'(x) = \frac{\lim_{h\to 0} g(x) \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} - \lim_{h\to 0} f(x) \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}}{\lim_{h\to 0} g(x) \lim_{h\to 0} g(x+h)}$$
$$= \frac{g(x)f'(x)-f(x)g'(x)}{\left[g(x)\right]^2}$$

Note that while the quotient rule can be proved from the product rule with $f(x)\left[g(x)\right]^{-1}$ it requires a result which has yet to be proved from first principles, namely the chain rule (see below).

E. The Chain Rule

Let $y = f(x)$. The corresponding increment in *y* given an increment in *x* by Δx is $\Delta y = f(x+\Delta x) - f(x)$.

Hence another way to define the derivative is:

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Now let $\varepsilon = \frac{\Delta y}{\Delta x} - f'(x)$. (Note that $\lim_{\Delta x \to 0} \varepsilon = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} - \lim_{\Delta x \to 0} f'(x) = 0$.)

Rearranging this gives:

$\Delta y = f'(x)\Delta x + \varepsilon \Delta x$ (provided the derivative of *f* exists)

Let $y = f(u), u = g(x)$. The corresponding increment in *u* and *y* given an increment in *x* by Δx is Δu and Δy respectively.

Applying the previous result:

$\Delta u = g'(x)\Delta x + \varepsilon_1 \Delta x = \left[g'(x) + \varepsilon_1\right]\Delta x$ where $\Delta x \to 0 \Rightarrow \varepsilon_1 \to 0$

$\Delta y = f'(u)\Delta u + \varepsilon_2 \Delta u = \left[f'(u) + \varepsilon_2\right]\Delta u$ where $\Delta u \to 0 \Rightarrow \varepsilon_2 \to 0$

Substituting the expression for Δu into Δy gives:

$$\Delta y = \left[f'(u) + \varepsilon_2\right]\left[g'(x) + \varepsilon_1\right]\Delta x \Rightarrow \frac{\Delta y}{\Delta x} = \left[f'(u) + \varepsilon_2\right]\left[g'(x) + \varepsilon_1\right]$$

Taking limits give the derivative for function of a function:

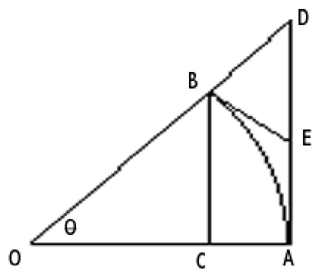
$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \\ &= \lim_{\Delta x \to 0} \left[f'(u) + \varepsilon_2\right]\left[g'(x) + \varepsilon_1\right] \\ &= f'(u)\, g'(x) \end{aligned}$$

The Chins Rule is often written in the form $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.

F. Derivatives of Trigonometric Functions

In order to find the derivative function for trigonometric functions, two important limits must first be established.

Consider a sector of a circle with center *O*, central angle θ and radius 1. (refer to diagram below) By definition of the radian measure, $\text{arc}\, AB = \theta$. Drop the altitude from *B* and let the point of intersection on *OA* be *C*. Note that $|BC| = |OB|\sin\theta = \sin\theta$, and that $|BC| < |AB|$ (which follows from Pythagoras). Clearly $|AB| < \text{arc}\, AB$, so $|BC| < \text{arc}\, AB \Rightarrow \sin\theta < \theta$.



Let the tangents at *A* and *B* intersect at *E*, from the diagram it can be seen that $\text{arc}\, AB < |AE| + |EB|$. So:

$$\begin{aligned}
\theta &= \text{arc}\, AB \\
&< |AE| + |EB| \\
&\le |AE| + |ED| \\
&= |AD| \\
&= |OA|\tan\theta \\
&= \tan\theta
\end{aligned}$$

Now $\sin\theta < \theta \Rightarrow \dfrac{\sin\theta}{\theta} < 1$ and $\theta < \tan\theta \Rightarrow \theta < \dfrac{\sin\theta}{\cos\theta} \Rightarrow \cos\theta < \dfrac{\sin\theta}{\theta}$.
Combining the two inequalities give:

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

Since the cosine function is continuous, $\lim_{x\to 0}\cos x = \cos 0 = 1$. But $\lim_{x\to 0} 1 = 1$ as well so by the squeeze theorem (if you are unfamiliar with this theorem please refer to "Analysis and Approximations") it follows that:

$$\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta} = 1$$

The function $(\sin\theta)/\theta$ is even therefore its right and left limits at zero must equal. Hence:

$$\lim_{x\to 0^+}\frac{\sin x}{x} = \lim_{x\to 0^-}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\sin x}{x} = 1$$

which is the first limit required. (Remember that trigonometric functions must be in radians!)

The second limit can be deduced algebraically as follows:

$$\begin{aligned}\lim_{x\to 0}\frac{\cos x-1}{x} &= \lim_{x\to 0}\left[\frac{\cos x-1}{x}\frac{\cos x+1}{\cos x+1}\right]\\ &= \lim_{x\to 0}\frac{\cos^2 x-1}{x(\cos x+1)}\\ &= \lim_{x\to 0}\frac{-\sin^2 x}{x(\cos x+1)}\\ &= -\lim_{x\to 0}\frac{\sin x}{x}\frac{\sin x}{\cos x+1}\\ &= -\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{\sin x}{\cos x+1}\\ &= -1\left(\frac{0}{1+1}\right)=0\end{aligned}$$

Derivative of the Sine Function

Let $f(x)=\sin x$, then $f'(x)=\cos x$.

Proof

Recall the double angle formula for the sine function:

$\sin(A+B)=\sin A\cos B+\cos A\sin B$

Applying this result in the derivative yields the two limits established above:

$$\begin{aligned}f'(x) &= \lim_{a\to 0}\frac{f(x+a)-f(x)}{a}\\ &= \lim_{a\to 0}\frac{\sin(x+a)-\sin x}{a}\\ &= \lim_{a\to 0}\frac{\sin x\cos a+\cos x\sin a-\sin x}{a}\\ &= \lim_{a\to 0}\left[\frac{\sin x\cos a-\sin x}{a}+\frac{\cos x\sin a}{a}\right]\\ &= \lim_{a\to 0}\left[\sin x\left(\frac{\cos a-1}{a}\right)+\cos x\left(\frac{\sin a}{a}\right)\right]\\ &= \lim_{a\to 0}\sin x\cdot\lim_{a\to 0}\left(\frac{\cos a-1}{a}\right)+\lim_{a\to 0}\cos x\cdot\lim_{a\to 0}\left(\frac{\sin a}{a}\right)\\ &= \sin x(0)+\cos x(1)\\ &= \cos x\end{aligned}$$

Note that $\lim_{x\to L}\sin x=\sin L, \lim_{x\to L}\cos x=\cos L$ follows from the definition of continuous functions.

Derivative of the Cosine Function

Let $f(x)=\cos x$, then $f'(x)=-\sin x$.

Proof
Recall the double angle formula for the cosine function:
$\cos(A+B)=\cos A\cos B-\sin A\sin B$

$$\begin{aligned}
f'(x)&=\lim_{a\to 0}\frac{\cos(x+a)-\cos x}{a}\\
&=\lim_{a\to 0}\frac{\cos x\cos a-\sin x\sin a-\cos x}{a}\\
&=\lim_{a\to 0}\left[\frac{\cos x\cos a-\cos x}{a}-\frac{\sin x\sin a}{a}\right]\\
&=\lim_{a\to 0}\left[\cos x\left(\frac{\cos a-1}{a}\right)-\sin x\left(\frac{\sin a}{a}\right)\right]\\
&=\lim_{a\to 0}\cos x\cdot\lim_{a\to 0}\frac{\cos a-1}{a}-\lim_{a\to 0}\sin x\cdot\lim_{a\to 0}\frac{\sin a}{a}\\
&=\cos x(0)-\sin x(1)\\
&=-\sin x
\end{aligned}$$

Derivative of the Tangent Function

If $f(x)=\tan x$, then $f'(x)=\sec^2 x$.

Proof
While the tangent function can be differentiated by using the definition of a derivative, it is much easier to use three results which have already been established; namely the derivative of sine and cosine and the quotient rule:

$$\begin{aligned}
\frac{d}{dx}(\tan x)&=\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)\\
&=\frac{\cos x\frac{d}{dx}(\sin x)-\sin x\frac{d}{dx}(\cos x)}{\cos^2 x}\\
&=\frac{\cos x(\cos x)-\sin x(-\sin x)}{\cos^2 x}\\
&=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}\\
&=\frac{1}{\cos^2 x}=\sec^2 x
\end{aligned}$$