Proofs of Selected Theorems on Analysis and Approximations

Theorem on Limits for Sequences

Let $\lim_{n\to\infty} a_n = a, \lim_{n\to\infty} b_n = b$.

(1) By definition, $\exists N_1, N_2$ such that $|a_n - a| < \frac{\varepsilon}{2}\ \ \forall n > N_1$ and $|b_n - b| < \frac{\varepsilon}{2}\ \ \forall n > N_2$ given any $\varepsilon > 0$. Now $|(a_n \pm b_n) - (a \pm b)| \le |a_n - a| + |b_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ $\forall n > \max(N_1, N_2)$. Hence $\lim_{n\to\infty}(a_n \pm b_n) = a \pm b$.

(2) $|a_n b_n - ab| \le |a_n||b_n - b| + |b||a_n - a| \le k|b_n - b| + |b||a_n - a|$ but by definition $\exists N_1, N_2$ such that $|a_n - a| < \frac{\varepsilon}{2|b|}\ \ \forall n > N_1$ and $|b_n - b| < \frac{\varepsilon}{2k}\ \ \forall n > N_2$ given any $\varepsilon > 0$. So $|a_n b_n - ab| \le k\frac{\varepsilon}{2k} + |b|\frac{\varepsilon}{2|b|} = \varepsilon\ \ \forall n > \max(N_1, N_2)$. Hence $\lim_{n\to\infty} a_n b_n = ab$.

(3) Given any $\varepsilon > 0$, $\exists N_1, N_2$ such that $|b_n - b| < \frac{|b|}{x}\ \ \forall n > N_1$ and $|b_n - b| < \frac{\varepsilon|b|^2}{2}\ \ \forall n > N_2$. Then $\left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{|b_n - b|}{|b_n||b|} < \frac{\varepsilon|b|^2}{|b||b|} = \varepsilon\ \ \forall n > \max(N_1, N_2)$. Hence $\lim_{n\to\infty}\frac{1}{b_n} = \frac{1}{b}$ and by limits for products $\lim_{n\to\infty} a_n\left(\frac{1}{b_n}\right) = \lim_{n\to\infty}\frac{a_n}{b_n} = a\left(\frac{1}{b}\right) = \frac{a}{b}$.

The Squeeze Theorem

Let $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ and $a_n \le b_n \le c_n\ \forall n$. Then $b_n - L \le |c_n - L|$ and $b_n - L \ge a_n - L \Rightarrow -(b_n - L) \le |a_n - L|$.

By definition of limit $\exists N_1, N_2$ such that $|c_n - L| < \varepsilon\ \ \forall n > N_1$ and $|a_n - L| < \varepsilon\ \ \forall n > N_2$ for any given $\varepsilon > 0$. So $\forall n > \max(N_1, N_2)$, $b_n - L < \varepsilon$ and $-(b_n - L) < \varepsilon \Rightarrow |b_n - L| < \varepsilon$.

Test for Divergence

Let $S_n = \sum_{i=1}^{n} a_i$. Then $a_n = S_n - S_{n-1}$. If the series converges to a limit S then $\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{n-1} = S$. Hence $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(S_n - S_{n-1}) = S - S = 0$.

The Comparison Test

Let $S_n = \sum_{i=1}^{n} a_i, T_n = \sum_{i=1}^{n} b_i$. Since $\sum b_n$ is convergent $T_\infty = T$. If $a_n \le b_n \ \forall n$, then $S_n \le T_n \ \forall n$. This means that $S_n \le T \ \forall n$. Given the series has positive terms only, it is increasing and bounded above and therefore convergent by the Monotonic Sequence Theorem.

The Limit Comparison Test

Let *m* and *M* be positive numbers such that $m < c < M$. Since $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ substituting gives $m < \frac{a_n}{b_n} < M \Rightarrow mb_n < a_n < Mb_n$ for large enough *n*.

Now if $\sum b_n$ converges, so does $\sum Mb_n = M\sum b_n$; thus $\sum a_n$ converges by the comparison test. Similarly if $\sum b_n$ diverges, so does $\sum mb_n = m\sum b_n$; thus $\sum a_n$ diverges by the comparison test.

Alternating Series Estimation Theorem

The truncation error in using S_n as an approximation as *S* is given by $|R_n| = |S - S_n| = a_{n+1} - a_{n+2} + a_{n+3} - a_{n+4} + \cdots$. Recombining the terms give $|S - S_n| = a_{n+1} - \left[(a_{n+2} - a_{n+3}) + (a_{n+4} - a_{n+5}) + \cdots\right]$.

The terms in the square brackets are all positive as $a_{n+1} \le a_n$ is a condition for convergence. Now when a positive term is subtracted from a_{n+1} it gives a number at most as large as a_{n+1} itself. Hence $|R_n| = |S - S_n| \le a_{n+1}$.

The Absolute Convergence Test

By definition $-|a_n| \le a_n \le |a_n| \Rightarrow 0 \le a_n + |a_n| \le 2|a_n|$. Let $b_n = a_n + |a_n|$, then $0 \le b_n \le 2|a_n|$. If $\sum a_n$ is absolutely convergent then $\sum b_n$ is also convergent by the comparison test.

Since $a_n = b_n - |a_n|$ and both $\sum b_n$ and $\sum |a_n|$ are convergent, $\sum a_n$ is also convergent.

Theorem on Convergence Values for Power Series

Suppose $\sum a_n x^n$ converges, then by test for divergence $\lim_{n\to\infty} a_n x^n = 0$. Setting $\varepsilon = 1$ in the definition for a limit gives $\left|a_n x^n - 0\right| < \varepsilon = 1$ for large enough *n*.

Now $\left|a_n x^n\right| = \left|\frac{a_n b^n x^n}{b^n}\right| = \left|a_n b^n\right|\left|\frac{x^n}{b^n}\right|$. Given that $\sum a_n b^n$ is convergent and that $|x| < |b|$, $\left|a_n x^n\right| < \left|a_n b^n\right|$ and hence $\sum a_n x^n$ is convergent for $|x| < |b|$.

The Mean Value Theorem for Derivatives

Define a new function $d(x)$ as the difference between the function $f(x)$ and the line segment between $\{(a, f(a)), (b, f(b))\}$.

The function $d(x) = f(x) - f(a) - \left(\frac{f(b) - f(a)}{b-a}\right)(x-a)$ is continuous on $[a,b]$ and differentiable on (a,b). Further, $d(a) = d(b)$ as $f(a) - f(a) - \left(\frac{f(b) - f(a)}{b-a}\right)(a-a) = 0 = f(b) - f(a) - \left(\frac{f(b) - f(a)}{b-a}\right)(b-a)$. So the conditions for Rolle's Theorem has been satisfied.

By Rolle's Theorem $\exists c \in (a,b)$ such that $d'(c) = 0$. Differentiating $d(x)$ with respect to *x* gives $d'(x) = f'(x) - \frac{f(b) - f(a)}{b-a}$. Substituting the result from Rolle's Theorem gives $0 = f'(c) - \frac{f(b) - f(a)}{b-a} \Rightarrow f'(c) = \frac{f(b) - f(a)}{b-a}$.

The Mean Value Theorem for Integrals

Let $F(x) = \int_a^x f(t)\,dt$ be a function which is continuous on $[a,b]$ and differentiable on (a,b). Then by the mean value theorem for derivatives $\exists c \in (a,b)$ such that $f(c) = \frac{F(b) - F(a)}{b-a}$. But $\int_a^b f(x)\,dx = F(b) - F(a)$ by the second form of the Fundamental Theorem of Calculus hence $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$ for some value of $c \in (a,b)$.