Analysis and Approximations

Infinite Sequences

A sequence is a list of terms in some definite order. This is usually denoted by $\{a_1, a_2, \ldots\}$. Sequences can also be defined as a mapping $a : □ \mapsto □$.

e.g. Consider $a_n = \frac{n}{n+1}$. The gap between 1 and the n^{th} term is given by $1 - a_n = 1 - \frac{n}{n+1} = \frac{1}{n+1}$. As $n \to \infty$, this gap tends to zero. In other words the gap between 1 and the sequence can be made arbitrarily small by choosing larger *n*. So the limit of the sequence is $\lim_{n\to\infty} \frac{n}{n+1} = 1$.

Convergence can be defined as follows:

(1) A sequence $\{a_n\}$ has a limit *L* if given any $\varepsilon > 0$ there exists a number *N* such that $|a_n - L| < \varepsilon$ for all *n* chosen where $n > N$. The sequence is said to be convergent.

(2) If no such limit exists, then the sequence diverges. (e.g. oscillating, periodic, shoots off to infinity)

Theorem on Limits for Sequences

Let $\{a_n\}, \{b_n\}$ be convergent sequences with limits *a* and *b* respectively, then:

(1) $\lim_{n\to\infty} (a_n \pm b_n) = a \pm b$

(2) $\lim_{n\to\infty} a_n b_n = ab$

(3) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$ (provided $b \neq 0$)

(4) $\lim_{n\to\infty} c a_n = ca$ (where *c* is a constant)

The Squeeze Theorem

Let $\{a_n\}, \{c_n\}$ be two convergent sequences both with limit *L*. If $a_n \le b_n \le c_n$ for all $n \ge n_0$, then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = L$.

Monotonic Sequence Theorem

Every bounded monotonic sequence converges.

Theorem on Sequence of Partial Sums

If $\lim_{n\to\infty} S_n = \sum_{n=1}^{\infty} a_n$ exists, the sequence of partial sums (i.e. $\{S_n\}$) is said to converge. This implies also that the series itself converges.

Theorem on Limits for Series

If $\sum a_n, \sum b_n$ are convergent series then:

(1) $\sum a_n \pm \sum b_n = \sum (a_n \pm b_n)$

(2) $\sum ca_n = c\sum a_n$ (where c is a constant)

Test for Divergence

If $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$. (Converse is not true in general.) Hence a series $\sum_{n=1}^{\infty} a_n$ is divergent if $\lim_{n\to\infty} a_n = c, c \neq 0$ or $\lim_{n\to\infty} a_n = \infty$.

The Comparison Test

Let $\sum a_n$ be a positive series. (i.e. $a_n \geq 0\ \forall n$) If there exists a convergent series $\sum b_n$ such that $a_n \leq b_n\ \forall n$ then $\sum a_n$ is also convergent.

The *p*-Series Test

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges if $p \leq 1$ and converges if $p > 1$.

The Limit Comparison Test

Suppose $\sum a_n, \sum b_n$ are positive series (i.e. $a_n \geq 0, b_n \geq 0\ \forall n$). One of the following three cases applies:

(1) If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$ (where c is a constant) then they both converge or both diverge.

(2) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges then $\sum a_n$ also converges.

(3) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges then $\sum a_n$ also diverges.

The Alternating Series Test

An alternating series is one where the terms alternate between positive and negative.

The alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ is convergent if it satisfies:

(1) $|a_{n+1}| \leq |a_n|$

(2) $\lim_{n \to \infty} a_n = 0$

Alternating Series Estimation Theorem

The truncation error in using the n^{th} partial sum S_n as an estimate for the limit S is defined as $R_n = S - S_n$.

If $S = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$ (the limit of an alternating series) then the magnitude of the truncation error is bounded by $|R_n| = |S - S_n| \leq a_{n+1}$.

The Integral Test

Suppose that $f(x)$ is a continuous, positive decreasing function on $[1, \infty)$ and $a_n = f(n)$. One of the two following cases applies:

(1) If $\int_1^{\infty} f(x)\,dx$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

(2) If $\int_1^{\infty} f(x)\,dx$ diverges then $\sum_{n=1}^{\infty} a_n$ diverges.

Further, $\int_1^{\infty} f(x)\,dx \leq \sum_{n=1}^{\infty} a_n \leq \int_1^{\infty} f(x)\,dx + a_1$.

Absolute Convergence

A series $\sum a_n$ is said to be absolutely convergent if the series of absolute values $\sum |a_n| = |a_1| + |a_2| + \cdots$ is convergent. A series which is convergent but not absolutely convergent is said to be conditionally convergent.

The Absolute Convergence Test

A series is convergent if it is absolutely convergent. Convergence does not imply absolute convergence in general, the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ being a good counter example. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ is convergent by alternating series test but $\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$ is divergent by the *p*-series test.

The Ratio Test

For the series $\sum_{n=1}^{\infty} a_n$, one of the three following cases applies:

(1) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = c < 1$ then the series converges.

(2) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = c > 1$ then the series diverges.

(3) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = c = 1$ then the test is inconclusive.

This is in fact a test for absolute convergence, but since absolute convergence implies convergence it is often used as a test for convergence.

Power Series

A power series is a series of the form $\sum_{n=0}^{\infty} c_n x^n$ (or more generally expressed as $\sum_{n=0}^{\infty} c_n (x-a)^n$). Power series cannot be manipulated algebraically until their convergence is shown as they are infinite.

The convergence of a power series depends on the value of *x*, and the range of values for which the series converges can be found by the ratio test. However, the individual boundary cases have to be considered separately as the ratio test is inconclusive when $c = 1$.

Theorem on Convergence Values for Power Series

If a power series $\sum a_n x^n$ converges when $x = b, b \neq 0$ then it will converge for all values of *x* such that $|x| < |b|$.

Radius and Interval of Convergence for Power Series

For any given power series $\sum c_n (x-a)^n$ one of the followings must be true:

(1) The series converges only when $x = a$.
(2) The series converges $\forall x \in \mathbb{R}$.
(3) The series converges if $|x-a| < R$ for some $R \in \mathbb{R}^+$ and diverges if $|x-a| > R$ (boundary cases need to be considered separately).

Note that this theorem rules out the possibility of the power series converging at a set of non-zero discrete points.

R is called the radius of convergence of the power series. The set of values for which the series converges is the interval of convergence, namely $I = (a-R, a+R)$.

Differentiation of Power Series

If the power series $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent on $I = (a-R, a+R)$, $R > 0$, then so is the series of its derivative $\sum_{n=1}^{\infty} n a_n x^{n-1}$.

Rolle's Theorem

Let $f(x)$ be a function such that:

(1) $f(x)$ is continuous on $[a,b]$
(2) $f(x)$ is differentiable on (a,b)
(3) $f(a) = f(b)$

Then there exists a number $c \in (a,b)$ such that $f'(c) = 0$. (Note that the number *c* may not be unique.)

The Mean Value Theorem for Derivatives

Let $f(x)$ be a function which is continuous on $[a,b]$ and differentiable on (a,b), then there exists a number $c \in (a,b)$ such that $f'(c) = \frac{f(b) - f(a)}{b-a}$. (Again, *c* may not be unique.)

The Mean Value Theorem for Integrals

Let $f(x)$ be a function which is continuous on $[a,b]$ and differentiable on (a,b), then there exists a number $c \in (a,b)$ such that $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$. Intuitively this means that $f(c)$ equals the "average value" for $f(x)$ in $[a,b]$.

Taylor's Theorem

If $f^{(n)}(x)$ is continuous on $[\alpha,\beta]$ and differentiable on (α,β), then there exists $c, w \in (a,x) \le (\alpha,\beta)$ such that $f(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)(x-a)^i}{i!} + R_n(x)$. The remainder term can take one of the two forms:

(1) $R_n(x) = \frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!}$ (Lagrange Form)

(2) $R_n(x) = \frac{f^{(n)}(x-w)^n(x-a)}{n!}$ (Cauchy Form)

The function $T_n(x) = f(a) + f'(a)(x-a) + \cdots + f^{(n)}(a)\frac{(x-a)^n}{n!}$ is called the n^{th} degree Taylor polynomial approximation to $f(x)$ at *a*.

The Alternative Form for Taylor's Theorem

Let $x = a + h$, *h* being a variable. Then Taylor's Theorem becomes a function in terms of *h*; $f(x) = f(a+h) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)h^n}{n!}$. If a polynomial approximation is used, the remainder term becomes $R_n(h) = \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(a + \lambda h),\ 0 < \lambda < 1$.

The Trapezium Rule

Let (a,b) be divided into *n* smaller and equally spaced intervals. Define $x_n = x_0 + nh = a + nh$ and $y_n = f(x_n)$ where $h = \frac{b-a}{n}$. In using the sum of the area of trapezium $\{(x_n, 0), (x_{n+1}, 0)(x_n, y_n), (x_{n+1}, y_{n+1})\}$ to approximate the area under a curve gives $\int_a^b f(x)\,dx \approx \frac{h}{2}(y_0 + 2(y_1 + y_2 + \cdots + y_{n-1}) + y_n)$.

Error Bound for Trapezium Rule

Let $f(x), f'(x), f''(x)$ be continuous on $[a,b]$. Define the function in $f(x_n)$ as $T_n(f) = \frac{h}{2}(y_0 + 2(y_1 + y_2 + \cdots + y_{n-1}) + y_n)$. If $|f''(x)| \le A \; \forall x \in [a,b]$ where *A* is a constant, then the error in using $T_n(f)$ as an approximation to the integral $\int_a^b f(x)\,dx$ is bounded by $|E_T| = \left| T_n(f) - \int_a^b f(x)\,dx \right| \le \frac{(b-a)h^2 A}{12} = \frac{h^3 nA}{12}$.

Simpson's Rule

For any three distinct points on $f(x)$ (corresponding to three distinct x_i's) there is a unique parabola which passes through all three points. Let (a,b) be divided into *n* smaller and equally spaced intervals. Define $x_n = x_0 + nh = a + nh$ and $y_n = f(x_n)$ where $h = \frac{b-a}{n}$. In using the area under parabola (passing though $\{(x_0, y_0), (x_1, y_1), (x_2, y_2)\}, \{(x_2, y_2), (x_3, y_3), (x_4, y_4)\}$ etc.) to approximate the area under a curve gives $\int_a^b f(x)\,dx \approx \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n)$.

Error Bound for Simpson's Rule

Let $f(x), f'(x), f''(x), f'''(x), f^{(4)}(x)$ be continuous on $[a,b]$. Define the function in $f(x_n)$ as $T_n(f) = \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 \cdots + 2y_{n-2} + 4y_{n-1} + y_n)$. If $|f^{(4)}(x)| \le k \; \forall x \in [a,b]$ where *k* is a constant, then the error in using $T_n(f)$ as an approximation to the integral $\int_a^b f(x)\,dx$ is bounded by $|E_S| \le \frac{(b-a)^5 k}{180n^4} = \frac{h^5 nk}{180}$.

Iterative Sequences

An iterative sequence is one where the n^{th} term is expressed as a function of the previous term. (i.e. $x_n = g\left(x_{n-1}\right)$) If the sequence converges, then $x_n \to X, x_{n-1} \to X$ as $n \to \infty$. So the limit is the solution of $X = g\left(X\right)$. (However a solution X is not always a limit of the sequence.)

Condition for Convergence of Iterative Sequences

An iterative sequence will converge to a limit x if $\left|g'(x)\right| < 1$ and for the initial point x_0 sufficiently close to x. The point x is called attractive if $\left|g'(x)\right| < 1$ and repellent if $\left|g'(x)\right| > 1$.

The minimum range from which x_0 can be chosen such that the sequence converges is the solution of the inequality $\left|g'(x)\right| < 1$.

The Newton-Raphson Method

A numerical solution to $f\left(x\right) = 0$ (provided it is differentiable) can be generated by the Newton-Raphson iteration $x_{n+1} = x_n - \dfrac{f\left(x_n\right)}{f'\left(x_n\right)}$.

Notice that if a limit x exists, then $g\left(x\right) = x - \dfrac{f\left(x\right)}{f'\left(x\right)}$. So by the quotient rule

$$g'(x) = 1 - \frac{\left(f'(x)\right)^2 - f\left(x\right)f''(x)}{\left(f'(x)\right)^2} = \frac{f\left(x\right)f''(x)}{\left(f'(x)\right)^2}.$$ Since $f\left(x\right) = 0$, $\left|g'(x)\right| = 0$.

Hence the Newton-Raphson method is a super-attractive process.

The Order of Convergence

Suppose the limit of an iterative sequence $\left\{x_n\right\}$ is a. Consider a new sequence $w_n = x_n - a$, which has iteration $w_{n+1} = h\left(w_n\right)$ and limit 0.

The maximum range from which x_0 can be chosen such that the sequence converges can be determined as follows:

(1) $w_{n+1} = cw_n, \left|c\right| < 1 \Rightarrow I = \mathbb{R}$ (order 1)

(2) $w_{n+1} = c{w_n}^2, c \in \mathbb{R} \Rightarrow I = \left(\dfrac{-1}{\left|c\right|}, \dfrac{1}{\left|c\right|}\right)$ (order 2)

(3) $w_{n+1} = c{w_n}^k, c \in \mathbb{R} \Rightarrow I = \left(-\sqrt[(k-1)]{\dfrac{1}{\left|c\right|}}, \sqrt[(k-1)]{\dfrac{1}{\left|c\right|}}\right)$ (order k)

For the original sequence $\{x_n\}$ this range becomes $I + a$.