Sequences and continuity

Stephen Huggett [*]

July 2002

Suppose the set of real numbers $0 < r \leq 1$ were countable. Then it would be possible to count them, which is to put them in a list, which might look like this, for example:

$$\begin{array}{ccl} r_1 & = & 0.\mathbf{1}0357627183\ldots \\ r_2 & = & 0.1\mathbf{4}329806115\ldots \\ r_3 & = & 0.02\mathbf{1}66095213\ldots \\ r_4 & = & 0.430\mathbf{0}5357779\ldots \\ r_5 & = & 0.9255\mathbf{0}489101\ldots \\ r_6 & = & 0.59210\mathbf{3}43297\ldots \\ \vdots & \vdots & \vdots \end{array}$$

The numbers on the diagonal are $1, 4, 1, 0, 0, 3, \ldots$. Now we construct a new number whose nth digit is different from the nth diagonal number. For example, we could choose it to be 1 whenever the diagonal digit is not 1, and 2 whenever the diagonal digit is 1. We get

$$x = 0.212111\ldots.$$

This number x cannot be in the above list! So no such list exists, and so there are uncountably many real numbers.

How do we probe the small-scale structure of $\mathbb{R}$? We first have to decide what "small-scale" actually means!

Definition The sequence $\{a_n\}$ is a **null sequence** if, for each positive number ϵ, there is an integer N such that

$$|a_n| < \epsilon, \qquad \text{for all} \quad n > N.$$

Monotone Convergence Theorem If the sequence $\{a_n\}$ is
EITHER increasing and bounded above
OR decreasing and bounded below
then $\{a_n\}$ is convergent.

[*] `http://homepage.mac.com/stephen_huggett/home.html`

Suppose $(x, y) \in \mathbb{R}^2 : 0 < x, y \leq 1$. For example:

$$x = 0.3 \quad 01 \quad 2 \quad 007 \quad 08 \ldots$$

$$y = 0.009 \quad 2 \quad 05 \quad 1 \quad 0008 \ldots$$

Define $z : 0 < z \leq 1$ by

$$z = 0.3 \quad 009 \quad 01 \quad 2 \quad 2 \quad 05 \quad 007 \quad 1 \quad 08 \quad 0008 \ldots$$

This mapping $(x, y) \longrightarrow z$ is one-one and onto.

So there is a one-one onto mapping between $\mathbb{R}^2$ and $\mathbb{R}$, thus destroying the concept of dimension! If we want to preserve this concept (which we usually do) then we need the mapping to be more than one-one and onto: we need it to be *continuous* with a continuous inverse.

Definition A function $f : A \longrightarrow \mathbb{R}$ is **continuous** at a $(a \in A)$ if, for each sequence $\{x_n\}$ in A such that $x_n \to a$,

$$f(x_n) \to f(a).$$

The function f is **continuous** (on A) if f is continuous at each $a \in A$.

Intermediate Value Theorem Let f be a continuous function on $[a, b]$, and let k be any number lying between $f(a)$ and $f(b)$. Then there exists a number c in $]a, b[$ such that

$$f(c) = k.$$

References

The "diagonal" argument that $\mathbb{R}$ is not countable is not in M203. This version comes from *The Emperor's New Mind* by Roger Penrose, Oxford University Press 1989.

The definition of a null sequence is in the Handbook on page 36, and in Analysis A Unit 2 on page 12.

The Monotone Convergence Theorem is in the Handbook on page 38, and in Analysis A Unit 2 on page 34.

The construction of a one-one onto mapping from $\mathbb{R}^2$ to $\mathbb{R}$ is not in M203. This version comes from *Proofs from the BOOK* by Martin Aigner and Günter Ziegler, Springer-Verlag 1998.

The definition of continuity is in the Handbook on page 41, and in Analysis A Unit 4 on page 14.

The Intermediate Value Theorem is in the Handbook on page 41, and in Analysis A Unit 4 on page 28.