1 Convergence

$A_N \to L$ or $\lim_{N\to\infty} A_N = L$ both mean:

$$\forall\epsilon > 0, \exists M, s.t. |A_N - L| < \epsilon, \forall N > M \tag{1}$$

This is read: "for all epsilon greater than zero, there exists an M such that the distance between the Nth term and L is less than epsilon for all N greater than M"

or

"no matter how close you get to L ($\forall\epsilon > 0$), eventually ($\exists M$) all the remaining terms ($\forall N > M$) will be closer ($|A_N - L| < \epsilon$)."

In general when writing a proof about convergence, you'll start with "$|A_N - L| < \epsilon$", isolate N, and end up with a "$\forall N > M$" statement.

1.1 Examples

Q: Show that $\frac{2N}{N+3} \to 2$.

A: This is a statement about convergence, so immediately use the definition, where $A_N = \frac{2N}{N+3}$ and $L = 2$. You want to show:

$\forall\epsilon > 0, \exists M, s.t. |\frac{2N}{N+3} - 2| < \epsilon, \forall N > M$

Starting with "$|\frac{2N}{N+3} - 2| < \epsilon$" youll want to create a "$N > stuff$" statement. This "$stuff$" will be the M.

$|\frac{2N}{N+3} - 2| < \epsilon$

$|\frac{2N}{N+3} \frac{2(N+3)}{N+3}| < \epsilon$

$|\frac{2N-(2N+6)}{N+3}| < \epsilon$

$|\frac{-6}{N+3}| < \epsilon$

$\frac{6}{N+3} < \epsilon$

$6 < N\epsilon + 3\epsilon$

$6 - 3\epsilon < N\epsilon$

$\frac{6-3\epsilon}{\epsilon} < N$

$M = \frac{6-3\epsilon}{\epsilon}$

Theres your proof. Obviously $\forall\epsilon > 0, \exists M$. You know because any $\epsilon > 0$ that you pick immediately gives you an M, so it must exist.

Q: Show that $\frac{1}{N} \to 3$.

A: This is a statement about convergence, so immediately use the definition, where $A_N = \frac{1}{N}$ and $L = 2$.

$\frac{1}{N}$ does not actually converge to 3, so this will be a disproof. Ill show that $\forall \epsilon > 0$, there does not exist M.

Once again start with $|A_N - L| < \epsilon$ and turn it into $N > M$.

$|\frac{1}{N} - 3| < \epsilon$

$3 - \frac{1}{N} < \epsilon$

$3 - \epsilon < \frac{1}{N}$

$N < \frac{1}{3-\epsilon}$

This is not a "$N > M$" statement. Therefore, it is not true that the sequence converges to 3. If you repeat this same example line for line using $L = 0$ you'll get:

$|\frac{1}{N}| < \epsilon$

$\frac{1}{N} < \epsilon$

$\frac{1}{\epsilon} < N$

This is a "$N > M$" statement (with $M = \frac{1}{\epsilon}$), so in fact $\frac{1}{N} \to 0$.