Complex Exponentials

Earlier, when you saw the complex plane, all the numbers were written in the form "A+Bi".

Here you'll begin to use the famous Euler equation (pronounced "oiler", as in "one who oils") which is:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

Examples:

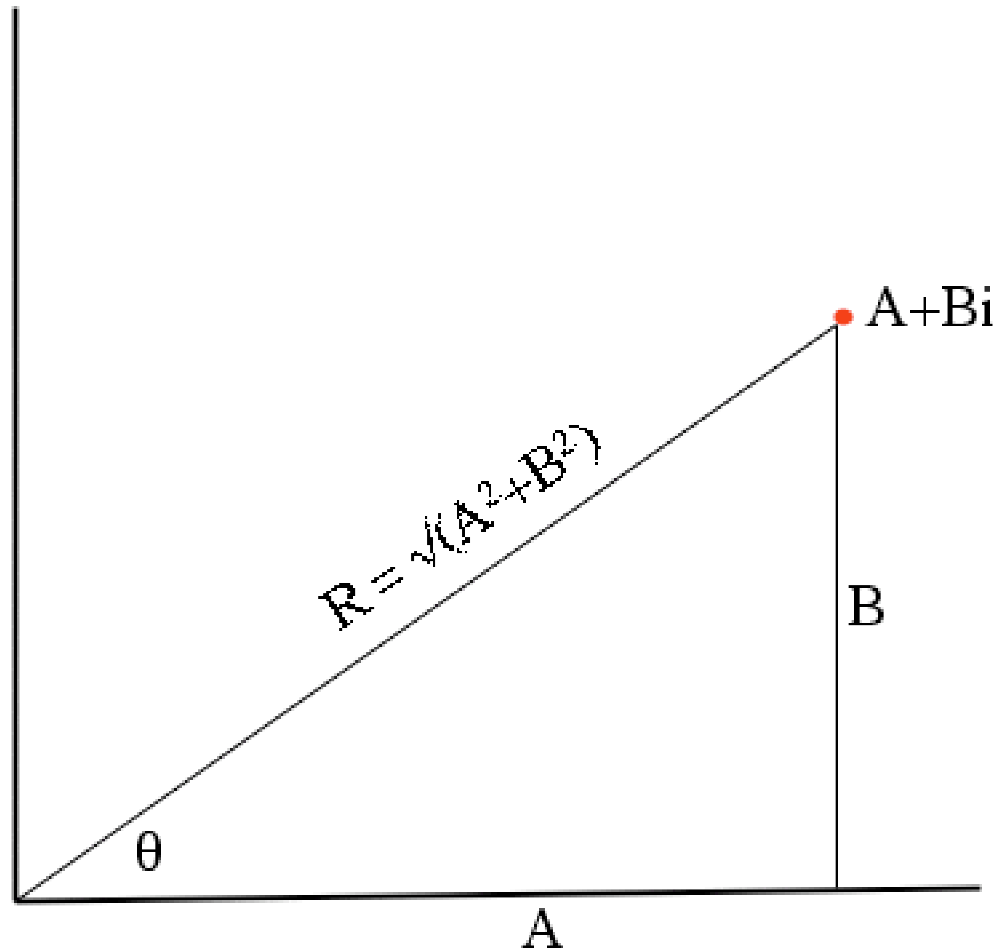
$e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1 + i0 = -1$

$e^{i\pi/4} = \cos(\pi/4) + i\sin(\pi/4)$

$= 1/\sqrt{(2)} + i/\sqrt{(2)}$

$e^{i5\pi/6} = \cos(5\pi/6) + i\sin(5\pi/6)$

$= -\sqrt{(3)}/2 + i/2$



Note that $A=R\cos(\theta)$ and $B=R\sin(\theta)$.

Converting $Re^{i\theta} \Rightarrow A+Bi$

(from the Euler equation)

$A = R\cos(\theta)$

$B = R\sin(\theta)$

Converting $A+Bi \Rightarrow Re^{i\theta}$

(from the picture above)

$R = \sqrt{(A^2+B^2)}$

$\theta = \text{Arctan}(B/A)$

Often it is more useful to write complex numbers as $Re^{i\theta}$ (= $R\cos(\theta)$ + $iR\sin(\theta)$), instead of as A+Bi.

	A+Bi	**$Re^{i\theta}$**
Add/Subtract	easy	difficult
Multiply	difficult	easy
Powers	difficult	easy
Roots	impossible	easy

Examples:

For these I'll use these two numbers

$J = 1 + i = \sqrt{(2)}e^{i\pi/4}$

$M = -2 + i2\sqrt{(3)} = 4e^{i2\pi/3}$

Addition:

A+Bi form:

$J+M = (1 + i) + (-2 + i2\sqrt{(3)})$

$= -1 + (1+2\sqrt{(3)})i$

$Re^{i\theta}$ form:

$J+M = \sqrt{(2)}e^{i\pi/4} + 4e^{i2\pi/3}$

Very difficult. This would involve vectors, the law of cosines, and the law of sines.

Multiplication:

A+Bi form:

$JM = (1 + i)(-2 + i2\sqrt{(3)})$

$= -2 - 2i + i2\sqrt{(3)} - 2\sqrt{(3)}$

$= (-2-2\sqrt{(3)}) + (2\sqrt{(3)}-2)i$

$Re^{i\theta}$ form:

$JM = (\sqrt{(2)}e^{i\pi/4})(4e^{i2\pi/3})$

$= 4\sqrt{(2)}e^{i\pi/4+i2\pi/3} = 4\sqrt{(2)}e^{i(\pi/4+2\pi/3)}$

$= 4\sqrt{(2)}e^{i11\pi/12}$

Powers:

A+Bi form:

$M^3 = (-2 + i2\sqrt{(3)})^3$

$= (-2+i2\sqrt{(3)})(-2+i2\sqrt{(3)})(-2+i2\sqrt{(3)})$

$= (4-i8\sqrt{(3)}+12i^2)(-2+i2\sqrt{(3)})$

$= (-8-i8\sqrt{(3)})(-2+i2\sqrt{(3)})$

$= 16-i16\sqrt{(3)}+i16\sqrt{(3)}-48i^2$

$= 16+48 = 64$

$Re^{i\theta}$ form:

$M^3 = (4e^{i2\pi/3})^3 = 4^3e^{i(2\pi/3)3} = 64e^{i2\pi} = 64$

Roots: (a quick aside)

There is a subtlety to finding roots. The Nth root of a number is written:

$\sqrt[N]{(Re^{i\theta})}$

Step 1: "Take the 1/N power"

$\sqrt[N]{(Re^{i\theta})} = (Re^{i\theta})^{1/N} = R^{1/N}e^{i\theta/N}$

There's your first root!

Step2: "Add $i2\pi/N$ to the angle N-1 times"

$R^{1/N}e^{i\theta/N + i2\pi/N}$ another root!

$R^{1/N}e^{i\theta/N + i2\pi/N + i2\pi/N}$ and another!

And so on...
The Nth time you add you'll get your first root (from step 1).

Example (that you've seen before):
$\sqrt{(9)} = ?$
Firstly, $9 = 9e^{i0}$
Step 1: "1/2 power"
$\sqrt{(9)} = (9e^{i0})^{1/2} = 9^{1/2}e^{i0/2} = 3e^{i0} = 3$
Step 2: "Add $i2\pi/2$ one time"
$3e^{i0 + i2\pi/2} = 3e^{i\pi} = 3(-1) = -3$

Roots:

A+Bi form:

$M^{1/4} = (-2 + i2\sqrt{(3)})^{1/4} = ?$

Nearly impossible

$Re^{i\theta}$ form:

$M^{1/4} = (4e^{i2\pi/3})^{1/4} = 4^{1/4}e^{i(2\pi/3)(1/4)}$

$= \sqrt{(2)}e^{i2\pi/12} = \sqrt{(2)}e^{i\pi/6}$

So the roots are:

$\sqrt{(2)}e^{i\pi/6}$

$\sqrt{(2)}e^{i\pi/6 + i2\pi/4} = \sqrt{(2)}e^{i2\pi/3}$

$\sqrt{(2)}e^{i\pi/6 + i2\pi/4 + i2\pi/4} = \sqrt{(2)}e^{i7\pi/6}$

$\sqrt{(2)}e^{i\pi/6 + i2\pi/4 + i2\pi/4 + i2\pi/4} = \sqrt{(2)}e^{i5\pi/3}$

Example:

$$(1+i) = \sqrt{(2)}e^{i\pi/4}$$

$$(1+i)^2 = 2e^{i2\pi/4} = 2e^{i\pi/2}$$

$$(1+i)^3 = 2\sqrt{(2)}e^{i3\pi/4}$$

$$(1+i)^4 = 4e^{i4\pi/4} = 4e^{i\pi}$$

$$(1+i)^5 = 4\sqrt{(2)}e^{i5\pi/4}$$

$$(1+i)^6 = 8e^{i6\pi/4} = 8e^{i3\pi/2}$$

$$(1+i)^7 = 8\sqrt{(2)}e^{i7\pi/4}$$