1 Complex Numbers

Complex Numbers were created to complete math on the real numbers. By using complex numbers you can finally take any root of any number, or completely factor an Nth degree polynomial into N pieces. Complex Numbers are called complex because it takes two real numbers to describe them.

$$i = \sqrt{-1}$$

or

$$i^2 = -1$$

From now on you can write $i\sqrt{3}$ instead of $\sqrt{-3}$. (3 is just an example) Any complex number Z can be written, $Z = A + Bi$, where A and B are real numbers. Real numbers are just complex numbers where $B = 0$. So $2 + 3i$, $-17 + \frac{13}{4}i$, $2i$, etc. are all complex numbers.

1.1 Two rules to rule them all

At the most basic level there are only two things you'll need to remember:

1) Treat i like a variable.

2) Except when you see i^2, since $i^2 = -1$.

Examples:

$(2 + 3i) + (-10 - i) = 2 + 3i - 10 - i = -8 + 2i$

$(2 + 3i)(-10 - i) = -2030i - 2i - 3i2 = -2030i - 2i + 3 = -17 - 32i$

$(5i) + (1 + 13i) = 5i + 1 + 13i = 1 + 18i$

$(5i)(1 + 13i) = 5i + 65i2 = -65 + 5i$

1.2 Complex Conjugates

If $Z = A + Bi$, then $\bar{Z} = A - Bi$. Notice that the complex conjugate of a real number does nothing. For example, $\bar{5} = \overline{5 + 0i} = 5 - 0i = 5$. Complex conjugates are useful primarily to "realify" complex numbers. $Z\bar{Z}$ is always a positive real number.

$$Z\bar{Z} = (A + Bi)(A - Bi) = A^2 + ABi - ABi - (Bi)^2 = A^2 - B^2 i^2 = A^2 + B^2 \tag{1}$$

You can use this in the context of complex division. For example:

$$\frac{7 + 2i}{-1 + 3i} = \frac{7 + 2i}{-1 + 3i}\frac{(-1 - 3i)}{(-1 - 3i)} = \frac{-7 - 2i + 21i + 6i^2}{(-1)^2 + 3^2} = \frac{-7 + 19i - 6}{10} = \frac{-13 + 19i}{10} = -\frac{13}{10} + \frac{19}{10}i \tag{2}$$

or for example:

$$\frac{1-3i}{2+i} = \frac{1-3i}{2+i}\frac{(2-i)}{(2-i)} = \frac{2-6i-i+3i^2}{2^2+1^2} = \frac{2-7i-3}{5} = \frac{-1-7i}{5} = -\frac{1}{5}-\frac{7}{5}i \tag{3}$$

The complex conjugate can also be used to find the "magnitude" of Z, which is just a more general term for "absolute value". Just like always, the absolute value is the distance to zero. Draw a picture of where $Z = A + Bi$ lies in the complex plane, and use the Pythagorean theorem. You'll find that $|Z| = \sqrt{A^2 + B^2}$. But you've already seen that $Z\bar{Z} = A^2 + B^2$, so

$$|Z| = \sqrt{Z\bar{Z}} \tag{4}$$

Finally, the conjugate is a very "transparent" operation. In the following take $Z = A + Bi$ and $Y = C + Di$.

$$\overline{Y+Z} = \overline{A+Bi+C+Di} = A - Bi + C - Di = \overline{A+Bi} + \overline{C+Di} = \bar{Y} + \bar{Z} \tag{5}$$

$$\overline{YZ} = \overline{(A+Bi)(C+Di)} = \overline{AC-BD+ADi+BCi} = AC-BD-ADi-BCi = (A-Bi)(C-Di) = \bar{Y}\bar{Z} \tag{6}$$

So the complex conjugate is transparent to addition and multiplication (and subtraction and division). As a result

$$\overline{Z^N} = \bar{Z}\bar{Z}...\bar{Z} = (\bar{Z})^N \tag{7}$$

Finally, if Z is a root of some polynomial $f(X) = A_N X^N + A_{N-1}X^{N-1} + ... + A_1 X + A_0$, where every A is a real number, then

$$0 = A_N Z^N + A_{N-1}Z^{N-1} + ... + A_1 Z + A_0 \tag{8}$$

$$\bar{0} = \overline{A_N Z^N + A_{N-1}Z^{N-1} + ... + A_1 Z + A_0} \tag{9}$$

$$0 = \overline{A_N Z^N} + \overline{A_{N-1}Z^{N-1}} + ... + \overline{A_1 Z} + \overline{A_0} \tag{10}$$

$$0 = A_N \overline{Z^N} + A_{N-1}\overline{Z^{N-1}} + ... + A_1 \bar{Z} + A_0 \tag{11}$$

$$0 = A_N \bar{Z}^N + A_{N-1}\bar{Z}^{N-1} + ... + A_1 \bar{Z} + A_0 \tag{12}$$

So if Z is a root, then so is $\bar{Z}$.

1.3 Nth Roots

When taking the Nth root of a number there are always N answers. Youve seen this before in the square root (2nd root), e.g. (4)1/2 = 2,-2. But for 3rd and higher roots youll need complex numbers to see all the solutions. In hw4 (and in a different way in an example below) you show that: $\sqrt[3]{1} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i, 1$

Example: Find all the solutions to $\sqrt[4]{16}$

So you're looking for solutions to $X^4 = 16$ This is the same as factoring $0 = X^4 - 16$ completely.

(Here keep in mind the fact that $A^2 - B^2 = (A + B)(A - B)$.)

$0 = X4 - 16$
$= (X2 - 4)(X2 + 4)$
$= (X - 2)(X + 2)(X^2 + 4)$
$= (X - 2)(X + 2)(X^2 - (2i)2)$
$= (X - 2)(X + 2)(X + 2i)(X - 2i)$
So $\sqrt[4]{16} = 2, -2, 2i, -2i$

Example: $\sqrt[3]{1} = ?$

This is the same as factoring out $0 = X^3 - 1$ completely. $X = 1$ is a solution. After dividing $X^3 - 1$ by $X - 1$ youll find: $X^3 - 1 = (X - 1)(X^2 + X + 1)$. Now use the quadratic equation to find the factors of $X^2 + X + 1$.

$X = \frac{-1 \pm \sqrt{(12-4(1)(1))}}{2(1)} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

So with these roots you have

$X^3 - 1 = (X - 1)(X - (-\frac{1}{2} + \frac{\sqrt{3}}{2}i))(X - (-\frac{1}{2} - \frac{\sqrt{3}}{2}i))$

And therefore:

$\sqrt[3]{1} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i, 1$