1 Vectors

A vector is a set of numbers that represent a point in space. In two dimensional space you write: $\vec{V} = (V_1, V_2)$, in three dimensions: $\vec{V} = (V_1, V_2, V_3)$, in four: $\vec{V} = (V_1, V_2, V_3, V_4)$, and so on. Two vectors are equal if and only if all of their terms $(V_1, V_2, ...)$ are equal. For this study guide I'll stick to three dimensions. In general I'll use $\vec{V}$ and $\vec{W}$ for vectors, and c and d for numbers.

There is four special vectors worth mentioning in advance.

$\vec{0} = (0, 0, 0)$. 0 is a number, but $\vec{0}$ is the origin.

$\hat{i} = (1, 0, 0)$

$\hat{j} = (0, 1, 0)$

$\hat{k} = (0, 0, 1)$

The length of a vector is found using the Pythagorean Theorem. For no good reason this length is called the "Magnitude" of the vector.

$$||\vec{V}|| = \sqrt{V_1^2 + V_2^2 + V_3^2} \tag{1}$$

Vectors can be added and subtracted. This is done term by term.

$$\vec{V} + \vec{W} = (V_1, V_2, V_3) + (W_1, W_2, W_3) = (V_1 + W_1, V_2 + W_2, V_3 + W_3) \tag{2}$$

The vector $\vec{V} + \vec{W}$ can be found visually by placing the tail of one vector at the head of the other.

The vector $(\vec{V} - \vec{W})$ is a vector that points in the direction of the line from $\vec{W}$ to $\vec{V}$.

Vectors can also be multiplied by a number, which is applied to each term. This also has the effect of multiplying the magnitude by the same amount.

$$c\vec{V} = c(V_1, V_2, V_3) = (cV_1, cV_2, cV_3) \tag{3}$$

$$||c\vec{V}|| = c||\vec{V}|| \tag{4}$$

$\hat{V}$ is the vector that points in the same direction as $\vec{V}$ but has length one. Vectors with length one are often called "unit vectors".

$$\hat{V} = \frac{\vec{V}}{||\vec{V}||} \tag{5}$$

There is no way to either add a vector to a number, nor multiply to vectors. This is like saying "2 + 5 miles north" or "up times left".

Instead of multiplication vectors have a "Dot Product" $(\vec{V} \cdot \vec{W})$ and a "Cross Product" $(\vec{V} \times \vec{W})$.

1.1 Dot Product

$$\vec{V}\cdot\vec{W} = (V_1, V_2, V_3)\cdot(W_1, W_2, W_3) = V_1W_1 + V_2W_2 + V_3W_3 \tag{6}$$

Here are the important things to know about the dot product:

$\vec{V}\cdot\vec{W} = ||\vec{V}||\,||\vec{W}||\cos\theta$ (Where θ is the angle between $\vec{V}$ and $\vec{W}$)
$\vec{V}\cdot\vec{W} = 0$ if and only if $\vec{V}\perp\vec{W}$.
$||\vec{V}||^2 = \vec{V}\cdot\vec{V}$
$\vec{V}\cdot\vec{W} = \vec{W}\cdot\vec{V}$
$\vec{U}\cdot(\vec{V}+\vec{W}) = \vec{U}\cdot\vec{V} + \vec{U}\cdot\vec{W}$
$(c\vec{V})\cdot\vec{W} = c(\vec{V}\cdot\vec{W})$

The dot product can be used to find what you get when you project one vector ($\vec{V}$) onto another ($\vec{W}$). This is written $P_{\vec{W}}(\vec{V})$.

$$P_{\vec{W}}(\vec{V}) = \left(\frac{\vec{V}\cdot\vec{W}}{\vec{W}\cdot\vec{W}}\right)\vec{W} \tag{7}$$

1.2 Cross Product

$$\vec{V}\times\vec{W} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ V_1 & V_2 & V_3 \\ W_1 & W_2 & W_3 \end{vmatrix} = (V_2W_3 - W_2V_3)\hat{i} + (W_1V_3 - V_1W_3)\hat{j} + (V_1W_2 - W_1V_2)\hat{k} \tag{8}$$

The cross product takes two vectors, and spits out a new vector that is perpendicular to both of the original vectors.

Notice that $\vec{V}\times\vec{W}$ is a <u>vector</u> while $\vec{V}\cdot\vec{W}$ is a <u>number</u>. Some important things to know about the cross product:

$||\vec{V}\times\vec{W}|| = ||\vec{V}||\,||\vec{W}||\sin\theta$
$\vec{V}\times\vec{W} = -\vec{W}\times\vec{V}$
$\vec{V}\times\vec{V} = 0$
$\vec{U}\times(\vec{V}+\vec{W}) = \vec{U}\times\vec{V} + \vec{U}\times\vec{W}$
$(c\vec{V})\times\vec{W} = c(\vec{V}\times\vec{W})$
$(\vec{V}\times\vec{W})\cdot\vec{V} = (\vec{V}\times\vec{W})\cdot\vec{W} = 0$ (By the definition of the Cross Product)
$(\vec{V}\times\vec{W})\cdot\vec{U} =$ the volume of the parallelepiped formed by $\vec{U},\vec{V},\vec{W}$

1.3 by the way...

Cross products and dot products can be generalized to higher dimensions. For example, in N dimensions:

$$\vec{V}\cdot\vec{W} = V_1W_1 + V_2W_2 + \cdots + V_NW_N \tag{9}$$

In two dimensions you'll only need one vector to find a perpendicular vector:

$$\begin{vmatrix} \hat{i} & \hat{j} \\ V_1 & V_2 \end{vmatrix} = V_2\hat{i} - V_1\hat{j} \tag{10}$$

In four dimensions you'd need three vectors to find a perpendicular vector.

$$(\vec{U} \times \vec{V} \times \vec{W}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} & \hat{l} \\ U_1 & U_2 & U_3 & U_4 \\ V_1 & V_2 & V_3 & V_4 \\ W_1 & W_2 & W_3 & W_4 \end{vmatrix} \tag{11}$$

In five dimensions you'd need 4 vectors and so on.