1 Taylor Polynomials

Almost every function you can think of has a Taylor polynomial. These polynomials are created by picking a value ($X = A$) and matching all the derivatives of the original function with all the derivatives of the Taylor polynomial at that value. These polynomials work perfectly inside of their "Radius of Convergence", and don't work at all outside.

Any means that you can come up with to find these polynomials will yield the same answer. Here are the two primary ways to do this:

2 Finding the Taylor Polynomial

2.1 Using a Previously Known Series

If you already know the polynomial for $F(x)$, then you can quickly find the polynomials for $F(2x)$, $F'(x)$, $F(x+1)$, $\int F(x)dx$, and so on. For the purposes of this class you'll need to know the geometric series:

$$\frac{1}{1-x} = \sum_{N=0}^{\infty} x^N$$

Q: What is the Taylor polynomial for $\frac{1}{x}$ about $A = 5$?
A:

$$\frac{1}{x} = \frac{1}{5-5+x} = \frac{1}{5-(5-x)} = \frac{1}{5}\frac{1}{1-(\frac{5-x}{5})} = \frac{1}{5}\sum_{N=0}^{\infty}\left(\frac{5-x}{5}\right)^N = \frac{1}{5}\sum_{N=0}^{\infty}\frac{(-1)^N}{5^N}(x-5)^N \tag{1}$$

Q: What is the Taylor polynomial for $\arctan(x)$ about $A = 0$?
A: Since $\int \frac{1}{x^2+1}dx = \arctan(x)$:

$$\arctan(x) = \int \frac{1}{x^2+1}dx = \int \frac{1}{1-(-x^2)}dx = \int \sum_{N=0}^{\infty}(-x^2)^N dx = \int \sum_{N=0}^{\infty}(-1)^N x^{2N} dx = \sum_{N=0}^{\infty}\frac{(-1)^N}{2N+1}x^{2N+1} \tag{2}$$

Q: Find the Taylor polynomial about $A = 0$ of $\ln(x+1)$.

A:

$$\ln(x+1) = \int \frac{1}{x+1}dx = \int \frac{1}{1-(-x)}dx = \int \frac{1}{1-(-x)}dx = \int \sum_{N=0}^{\infty}(-x)^N dx = -\sum_{N=0}^{\infty} \frac{(-x)^{N+1}}{N+1} \tag{3}$$

$$= -\sum_{N=1}^{\infty} \frac{(-x)^N}{N} = \sum_{N=1}^{\infty} \frac{(-1)^{N+1}}{N} x^N \tag{4}$$

Q: Find the Taylor polynomial about 1 of $F(x) = \frac{x}{x^2+5x+6}$.
A: $\frac{x}{x^2+5x+6} = \frac{x}{(x+2)(x+3)} = \frac{A}{(x+2)} + \frac{B}{(x+3)}$
$x = A(x+3) + B(x+2) = Ax + 3A + Bx + 2B = (A+B)x + (3A+2B)$
Matching terms leaves the equations $A + B = 1$ and $3A + 2B = 0$. So $A = -2$, $B = 3$. Therefore:

$$\frac{x}{x^2+5x+6} = \frac{-2}{x+2} + \frac{3}{x+3} = \frac{-2}{3-(1-x)} + \frac{3}{4-(1-x)} = \left(\frac{-2}{3}\right)\frac{1}{1-(\frac{1-x}{3})} + \left(\frac{3}{4}\right)\frac{1}{1-(\frac{1-x}{4})} \tag{5}$$

$$= \frac{-2}{3}\sum_{N=0}^{\infty}\left(\frac{1-x}{3}\right)^N + \frac{3}{4}\sum_{N=0}^{\infty}\left(\frac{1-x}{4}\right)^N = \frac{-2}{3}\sum_{N=0}^{\infty}\left(\frac{-1}{3}\right)^N (x-1)^N + \frac{3}{4}\sum_{N=0}^{\infty}\left(\frac{-1}{4}\right)^N (x-1)^N \tag{6}$$

$$= \sum_{N=0}^{\infty}\left(\frac{-2}{3}\left(\frac{-1}{3}\right)^N + \frac{3}{4}\left(\frac{-1}{4}\right)^N\right)(x-1)^N \tag{7}$$

2.2 Using the Definition of the Taylor Series

$$f(x) = \sum_{N=0}^{\infty} \frac{f^{(N)}(A)}{N!}(x-A)^N \tag{8}$$

Where $f^{(N)}(x)$ is the Nth derivative of $f(x)$ at $x = A$, $N! = 1 \cdot 2 \cdot 3 \cdots (N-1) \cdot N$, and A is the x value being expanded about.

Q: What is the Taylor polynomial for $\frac{1}{x}$ about $A = 5$?
A: $f(x) = x^{-1}$ $f(5) = \frac{1}{5}$
$f'(x) = -x^{-2}$ $f'(5) = -\frac{1}{5^2}$
$f''(x) = 2x^{-3}$ $f''(5) = \frac{2}{5^3}$
$f^{(3)}(x) = -2 \cdot 3x^{-4}$ $f^{(3)}(5) = -\frac{2\cdot 3}{5^4}$
$f^{(4)}(x) = 2 \cdot 3 \cdot 4x^{-5}$ $f^{(4)}(5) = \frac{2\cdot 3\cdot 4}{5^5}$
Following this pattern you should see:

$f^{(N)}(5) = \frac{(-1)^N N!}{5^{N+1}}$, for all N.

(Compare with the definition of the Taylor Polynomial)

$$\frac{1}{x} = \sum_{N=0}^{\infty} \frac{(-1)^N N!}{5^{N+1} N!}(x-5)^N = \sum_{N=0}^{\infty} \frac{(-1)^N}{5^{N+1}}(x-5)^N = \frac{1}{5}\sum_{N=0}^{\infty} \frac{(-1)^N}{5^N}(x-5)^N \tag{9}$$

Q: Find the Taylor polynomial about $A = 0$ of $\ln(x+1)$.

A: $f(x) = \ln(x+1)$ $f(0) = \ln(1)$

$f'(x) = (x+1)^{-1}$ $f'(0) = 1$

$f''(x) = -(x+1)^{-2}$ $f''(0) = -1$

$f^{(3)}(x) = 2(x+1)^{-3}$ $f^{(3)}(0) = 2$

$f^{(4)}(x) = -2 \cdot 3(x+1)^{-4}$ $f^{(4)}(0) = -2 \cdot 3$

$f^{(5)}(x) = 2 \cdot 3 \cdot 4(x+1)^{-5}$ $f^{(5)}(0) = 2 \cdot 3 \cdot 4$

Following this pattern you should see:

$f^{(N)}(0) = (-1)^{N+1}(N-1)!$, for $N1$.

(Compare with the definition of the Taylor Polynomial, and leave the 0th term out since the pattern doesn't fit for $N = 0$)

$$\ln(x+1) = \ln 1 + \sum_{N=1}^{\infty} \frac{(-1)^{N+1}(N-1)!}{N!}x^N = \sum_{N=1}^{\infty} \frac{(-1)^{N+1}}{N}x^N \tag{10}$$

Q: Find the Taylor polynomial about $A = 0$ of $\cos(x)$.

A: $f(x) = \cos(x)$ $f(0) = 1$

$f'(x) = -\sin(x)$ $f'(0) = 0$

$f''(x) = -\cos(x)$ $f''(0) = -1$

$f^{(3)}(x) = \sin(x)$ $f^{(3)}(0) = 0$

$f^{(4)}(x) = \cos(x)$ $f^{(4)}(0) = 1$

Following this pattern you should see:

$$f^{(N)}(0) = \begin{cases} (-1)^{N/2} & \text{for N even} \\ 0 & \text{for N odd} \end{cases} \tag{11}$$

or

$$f^{(2N)}(0) = (-1)^N \tag{12}$$

Comparing with the definition of the Taylor Polynomial, and then doing an index shift $(N \to 2N)$

$$\cos(x) = \sum_{N=0}^{\infty} \frac{f^{(N)}(0)}{N!}x^N = \sum_{N=0}^{\infty} \frac{f^{(2N)}(0)}{(2N)!}x^{2N} = \sum_{N=0}^{\infty} \frac{(-1)^N}{(2N)!}x^{2N} \tag{13}$$

3 Derivatives

When taking the derivative you can go term by term.

$$\left(\sum_{N=0}^{\infty} A_N X^N\right)' = \sum_{N=0}^{\infty} N A_N X^{N-1} = \sum_{N=1}^{\infty} N A_N X^{N-1} = \sum_{N=0}^{\infty} (N+1) A_{N+1} X^N \tag{14}$$

Notice that since the N=0 term is constant it goes away.

Q: Show that $(\cos(x))'' = -\cos(x)$.
A:

$$(\cos(x))'' = \left(\sum_{N=0}^{\infty} \frac{(-1)^N}{(2N)!} x^{2N}\right)'' = \left(\sum_{N=0}^{\infty} \frac{(-1)^N 2N}{(2N)!} x^{2N-1}\right)' = \left(\sum_{N=1}^{\infty} \frac{(-1)^N 2N}{(2N)!} x^{2N-1}\right)' \tag{15}$$

$$= \sum_{N=1}^{\infty} \frac{(-1)^N 2N(2N-1)}{(2N)!} x^{2N-2} = \sum_{N=1}^{\infty} \frac{(-1)^N}{(2N-2)!} x^{2N-2} = \sum_{N=0}^{\infty} \frac{(-1)^{N+1}}{(2N)!} x^{2N} \tag{16}$$

$$= -\sum_{N=0}^{\infty} \frac{(-1)^N}{(2N)!} x^{2N} = -\cos(x) \tag{17}$$