1 Lines and Curves

A curve is a function that turns a single variable (usually "time") and turns it into a position in space.

A line is a curve that doesn't curve. It runs in one direction. Lines can be defined by a direction ($\vec{V}$), and point ($\vec{P}$) on the line.

$\vec{L}(t) = \vec{P} + t\vec{V}$

Q: Find the line that runs through the points $(1, 2, 3)$ and $(0, 1, -1)$.

A: The direction that the line is going is given by $\vec{V} = (1, 2, 3) - (0, 1, -1) = (1, 1, 4)$. Pick a point: say $\vec{P} = (1, 2, 3)$.

So, $\vec{L}(t) = (1, 2, 3) + t(1, 1, 4) = (t + 1, t + 2, 4t + 3)$.

If I had chosen $\vec{P} = (0, 1, -1)$ or any other point on the line, I'd have gotten the same line, just parameterized differently. This is akin to taking the same road trip that a friend does, but leaving a day before or after.

Q: Where do the lines $\vec{C}(t) = (-2t, 3t + 2, 2 + t)$ and $\vec{D}(s) = (2 + 7s, -s - 1, 2s + 1)$ intersect?

A: "Intersection" means that the x, y, and z values of both lines are the same.

$$\begin{aligned} -2t = 2 + 7s &\quad\rightarrow\quad t = -1 - \tfrac{7}{2}s \\ 3t + 2 = -s - 1 &\quad\rightarrow\quad s = -3t - 3 \\ 2 + t = 2s + 1 &\quad\rightarrow\quad t = 2s - 1 \end{aligned}$$

Plugging the third equation into the second:

$s = -3(2s - 1) - 3 = -6s + 3 - 3 = -6s \rightarrow 7s = 0 \rightarrow s = 0$

Therefore: $t = -1$ (from either the first or third equation)

The intersection is at $\vec{C}(-1) = \vec{D}(0) = (2, -1, 1)$.

Q: Where do the lines $\vec{C}(t) = (t, 2t - 1, 4 - t)$ and $\vec{D}(s) = (1 - s, s + 2, 3s - 1)$ intersect?

A:

$$\begin{aligned} t = 1 - s &\quad\rightarrow\quad t = 1 - s \\ 2t - 1 = s + 2 &\quad\rightarrow\quad s = 2t - 3 \\ 4 - t = 3s - 1 &\quad\rightarrow\quad t = 5 - 3s \end{aligned}$$

Plugging the first equation into the second:

$s = 2(1 - s) - 3 = 2 - 2s - 3 = -2s - 1 \rightarrow 3s = -1 \rightarrow s = -\frac{1}{3}$

Now from the first equation you get $t = \frac{4}{3}$, but from the third equation you get $t = 6$.

Therefore you're left with a contradiction. When that happens it means that your original assumption was incorrect. (The assumption being that the lines <u>do</u> intersect)

These lines are "skew" lines. They never intersect.

1.1 curves and velocity

Curves are a generalization of lines. For lines the equations for x, y, and z are linear. For curves the equations can be anything. For example:

$\vec{C}(t) = (t^2, t, \sin(t))$

$\vec{C}(t) = (0, \sqrt{t+5}, |t|)$

$\vec{V}(t) = \vec{C}'(t)$

Velocity completely describes how a point is moving. $\hat{V}$ is the direction the point is moving in, and $|\vec{V}|$ is the speed of the movement.

One of the most important things about velocity is that it automatically yields the tangent vector a curve.

Q: Find the line $\vec{L}(t)$ that is tangent to the curve $\vec{C}(t) = (t, t^2, t^3)$ at $t = 1$.

A: To find a line you need a direction and a point on the line. The point will be $\vec{C}(1) = (1, 1, 1)$. The direction will be $\vec{C}'(1)$.

$\vec{V}(t) = \vec{C}'(t) = (1, 2t, 3t^2)$

$\vec{C}'(1) = (1, 2, 3)$

Therefore $\vec{L}(t) = (1, 1, 1) + t(1, 2, 3) = (1 + t, 1 + 2t, 1 + 3t)$.

Q: A fly on the rim of a wheel (radius 1) that is rolling to the right, is moving forward ($\vec{C}_{forward}(t) = (t, 1)$) and turning clockwise ($\vec{C}_{turning}(t) = (-\cos(t), \sin(t))$).

$\vec{C}(t) = (t - \cos(t), 1 + \sin(t)))$

When is the fly moving the fastest/slowest?

A: $\vec{V}(t) = \vec{C}'(t) = (1 + \sin(t), \cos(t)))$

Speed is given by $|\vec{V}|$, so:

$|\vec{C}'(t)| = \sqrt{(1 + \sin(t))^2 + (\cos(t))^2} = \sqrt{1 + 2\sin(t) + (\sin(t))^2 + (\cos(t))^2} = \sqrt{2 + 2\sin(t)}$

$|\vec{C}'(t)| = 0$ (stationary) at $t = 3\pi/2, 7\pi/2, 11\pi/2, ...$ At these times the fly is at ground level (where the rubber meets the road, which is not moving).

$|\vec{C}'(t)| = \sqrt{4} = 2$ at $t = \pi/2, 5\pi/2, 9\pi/2, ...$ At these times the fly is at the top of the wheel, where the forward motion combines with the rotational motion of the wheel.

1.2 acceleration

Acceleration comes in two flavors:

$\vec{A_T}$ = "**Tangential Acceleration**" is acceleration in the direction of movement. It speeds you up and slows you down. You feel $\vec{A_T}$ when you step on the gas or the brake.

$\vec{A_N}$ = "**Normal Acceleration**" is acceleration perpendicular to the direction of movement. It causes you to change direction. $\vec{A_N}$ is usually called "centrifugal acceleration".

$\vec{C}(t)$ = position
$\vec{V}(t) = \vec{C}'(t)$ = velocity
$\vec{A}(t) = \vec{V}'(t) = \vec{C}''(t)$ = acceleration
$\vec{A}(t) = \vec{A_T}(t) + \vec{A_N}(t)$

Since $\vec{A_T}(t)$ points in the direction of movement it is by definition the part of $\vec{A}(t)$ that points in the direction of $\vec{V}(t)$ (the projection of $\vec{A}(t)$ onto $\vec{V}(t)$).

$$\vec{A_T} = \vec{P_{\vec{V}}}(\vec{A}) = \left(\frac{\vec{A} \cdot \vec{V}}{\vec{V} \cdot \vec{V}}\right)\vec{V} \quad \text{and} \quad \vec{A_N} = \vec{A} - \vec{A_T} \tag{1}$$

In the book this is written:

$$\vec{A_T} = |\vec{V}'|\vec{T} \quad \text{and} \quad \vec{A_N} = \kappa|\vec{V}|^2\vec{N} \tag{2}$$

Where $\vec{T}$ is the unit tangent vector to the curve, $\vec{N}$ is perpendicular and pointing inward, and κ is the curvature in the next section.

Q: Find the centrifugal acceleration of a particle traveling in a circle of radius R, at speed S.

A: First write a general equation for a circle of radius R.
$\vec{C}(t) = (R\cos(\lambda t), R\sin(\lambda t))$, where λ controls the speed but is so far unknown.
$\vec{V}(t) = \vec{C}'(t) = (-R\lambda\sin(\lambda t), R\lambda\cos(\lambda t))$
$\vec{A}(t) = \vec{C}''(t) = (-R\lambda^2\cos(\lambda t), -R\lambda^2\sin(\lambda t))$
$S = |\vec{V}(t)| = R\lambda \rightarrow \lambda = \frac{S}{R}$ Using this value of λ in the equation for $\vec{A}(t)$:
$\vec{A}(t) = \vec{C}''(t) = (-\frac{S^2}{R}\cos(\frac{S}{R}t), -\frac{S^2}{R}\sin(\frac{S}{R}t))$

Notice that $\vec{A}$ always points in exactly the opposite direction as $\vec{C}$. This means that the acceleration on one side of the circle is always towards the other side.
$|\vec{A}(t)| = \frac{S^2}{R}$

1.3 Curvature

The curvature, $\kappa(t)$, of a curve, $\vec{C}(t)$. is a measure of how sharply $\vec{C}(t)$ is turning a corner. The sharper the turn, the higher the curvature.

Visually, if you can fit a circle or radius R to the curve. Then $\kappa = \frac{1}{R}$. For obscure reasons (covered in class) this is the simplest equation that works in general:

$$\kappa(t) = \frac{|\vec{C}''(t) \times \vec{C}'(t)|}{|\vec{C}'(t)|^3} \tag{3}$$

Q: Find the curvature of the helix $\vec{C}(t) = (\cos(t), \sin(t), t)$.
A:
$\vec{C}'(t) = (-\sin(t), \cos(t), 1)$
$|\vec{C}'(t)| = \sqrt{\sin^2(t) + \cos^2(t) + 1} = \sqrt{2}$
$\vec{C}''(t) = (-\cos(t), -\sin(t), 0)$

$$\vec{C}''(t) \times \vec{C}'(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\cos(t) & -\sin(t) & 0 \\ -\sin(t) & \cos(t) & 1 \end{vmatrix}$$

$$= (-\sin(t)\hat{i} + 0\hat{j} - \cos^2(t)\hat{k}) - (0\hat{i} - \cos(t)\hat{j} + \sin^2(t)\hat{k})$$

$$= -\sin(t)\hat{i} + \cos(t)\hat{j} - (\cos^2(t) + \sin^2(t))\hat{k} = (-\sin(t), \cos(t), -1)$$

$|\vec{C}''(t) \times \vec{C}'(t)| = \sqrt{\sin^2(t) + \cos^2(t) + 1} = \sqrt{2}$
$\kappa(t) = \frac{|\vec{C}''(t) \times \vec{C}'(t)|}{|\vec{C}'(t)|^3} = \frac{\sqrt{2}}{(\sqrt{2})^3} = \frac{1}{2}$

Q: Find the curvature of the line $\vec{C}(t) = (2 + t, -3 + 2t, 19)$.
A: You should guess that $\kappa(t) = 0$, but here's how you prove it.
$\vec{C}'(t) = (1, 2, 0)$
$|\vec{C}'(t)| = \sqrt{5}$
$\vec{C}''(t) = (0, 0, 0)$
$\vec{C}''(t) \times \vec{C}'(t) = (0, 0, 0)$
$|\vec{C}''(t) \times \vec{C}'(t)| = 0$
$\kappa(t) = \frac{|\vec{C}''(t) \times \vec{C}'(t)|}{|\vec{C}'(t)|^3} = \frac{0}{(\sqrt{5})^3} = 0$

2 Planes

A plane in 3 dimensions is completely defined by a perpendicular vector, and a point in the plane.

For example the ceiling is defined by the normal ("normal" = "perpendicular") vector $\hat{k}$ (which points up), and a light bulb (which is in the ceiling. The floor is also a plane defined by $\hat{k}$, and your foot (a point on the ground).

Any vector that points along the plane will be perpendicular to the normal vector $\vec{n}$. So here's the trick: Say you already have a point, $\vec{P}$, in the plane.

If $\vec{W} = (x, y, z)$ is in the plane, then the vector that points between $\vec{P}$ and $\vec{W}$ (which is $\vec{W} - \vec{P}$) will be parallel to the plane, and thus perpendicular to $\vec{n}$.

You can write this mathematically as:

$$0 = \vec{n} \cdot (\vec{W} - \vec{P}) \tag{4}$$

or if $\vec{n} = (A, B, C)$ and $\vec{n} \cdot \vec{P} = D$:

$$D = \vec{n} \cdot \vec{P} = \vec{n} \cdot \vec{W} = (A, B, C) \cdot (x, y, z) = Ax + By + Cz \tag{5}$$

Q: Find the plane perpendicular to $(1, 2, -5)$ and containing the point $(2, 0, -1)$.

A:

$0 = (1, 2, -5) \cdot ((x, y, z) - (2, 0, -1))$

$0 = x + 2y - 5z - 2 - 5$

$7 = x + 2y - 5z$

Q: Find the plane that contains the lines $C(t) = (t + 1, t - 5, 1 - t)$ and $D(t) = (3t - 1, -4t, 0)$.

A: $C(t) = (1, -5, 1) + t(1, 1, -1)$ and $D(t) = (-1, 0, 0) + t(3, -4, 0)$

The normal vector will be perpendicular to the direction of both lines.

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -4 & 0 \\ 1 & 1 & -1 \end{vmatrix} = (4\hat{i} + 0\hat{j} + 3\hat{k}+) - (0\hat{i} - 3\hat{j} - 4\hat{k}) = 4\hat{i} + 3\hat{j} + 7\hat{k} = (4, 3, 7)$$

Any point on either line will work as a sample point in the plane. In this case $\vec{P} = (-1, 0, 0)$.

$0 = \vec{n} \cdot ((x, y, z) - \vec{P}) = (4, 3, 7) \cdot (x + 1, y, z) = 4x + 4 + 3y + 7z$

$-4 = 4x + 3y + 7z$