1 Sequences

$[A_N]$ is a sequence of numbers A_1, A_2, A_3, ...

$[A_N]$ is called "monotonic increasing" if $A_N \leq A_{N+1}$ for all N.

$[A_N]$ is called "monotonic decreasing" if $A_N \geq A_{N+1}$ for all N.

$[A_N]$ is called convergent if $A_N \longrightarrow L$.

$[A_N]$ is called divergent otherwise.

1.1 L'Hospital

L'Hospital can be used for continuous functions, and any sequence that can be expressed as a continuous function ($f(N) = A_N$).

If $\lim\limits_{x\to L} f(x) = \lim\limits_{x\to L} g(x) = 0$ or $\pm\infty$, then:

$$\lim_{x\to L} \frac{f(x)}{g(x)} = \lim_{x\to L} \frac{f'(x)}{g'(x)} \tag{1}$$

Example:

$$\lim_{x\to 0} \frac{Sin(x)}{x} = \lim_{x\to 0} \frac{Cos(x)}{1} = 1 \tag{2}$$

1.2 Squeeze Theorem

If $A_N \leq B_N \leq C_N$, then $\lim\limits_{N\to\infty} A_N \leq \lim\limits_{N\to\infty} B_N \leq \lim\limits_{N\to\infty} C_N$

In particular if $\lim\limits_{N\to\infty} A_N = \lim\limits_{N\to\infty} C_N = L$, then $\lim\limits_{N\to\infty} B_N = L$.

Example:

$A_N = 0$, $B_N = \frac{1+(-1)^N}{N}$, $C_N = \frac{2}{N}$

$\lim\limits_{N\to\infty} 0 = \lim\limits_{N\to\infty} \frac{2}{N} = 0$, therefore $\lim\limits_{N\to\infty} \frac{1+(-1)^N}{N} = 0$

2 Series

Series are sums of numbers.

The sum of the first M terms is written: $S_M = \sum\limits_{N=0}^{M} A_N$

An infinite sum is written: $S = \sum\limits_{N=0}^{\infty} A_N = \lim\limits_{M\to\infty} \sum\limits_{N=0}^{M} A_N$

If $S = \sum\limits_{N=0}^{\infty} A_N = L \neq \infty$, then S is called "convergent".

If S is not equal to some finite number, then S is called "divergent". Example:

$$S = \sum_{N=0}^{\infty} N = \infty \Rightarrow S \text{ is divergent.}$$
$$S = \sum_{N=0}^{\infty} \sin N \text{ never settles down to and particular number, so } S \text{ is divergent.}$$

If $\lim_{N\to\infty} A_N \neq 0$, then $S = \sum_{N=0}^{\infty} A_N = \infty$

2.1 Geometric Series

$$\sum_{N=0}^{\infty} R^N = \begin{cases} \frac{1}{1-R} & : |R| < 1 \\ divergent & : |R| \geq 1 \end{cases}$$

2.2 Integral Test

If $[A_N]$ is monotonically decreasing, and there is a function $f(x)$ such that $f(N) = A_N$ then:

$$\sum_{N=0}^{\infty} A_N = \sum_{N=0}^{\infty} f(N) < \infty \text{ if and only if} \int_B^{\infty} f(x)dx < \infty \tag{3}$$

2.3 P Test

L is a (non-infinite) number.

$$\sum_{N=1}^{\infty} \frac{1}{N^P} = \begin{cases} divergent & : P \leq 1 \\ convergent & : P > 1 \end{cases}$$

2.4 Ratio Test

$$\sum_{N=0}^{\infty} A_N = \begin{cases} divergent & : \lim_{N\to\infty} \frac{A_{N+1}}{A_N} > 1 \\ convergent & : \lim_{N\to\infty} \frac{A_{N+1}}{A_N} < 1 \\ inconclusive & : \lim_{N\to\infty} \frac{A_{N+1}}{A_N} = 1 \end{cases}$$

2.5 Root Test

$$\sum_{N=0}^{\infty} A_N = \begin{cases} divergent & : \lim_{N\to\infty} \sqrt[N]{|A_N|} > 1 \\ convergent & : \lim_{N\to\infty} \sqrt[N]{|A_N|} < 1 \\ inconclusive & : \lim_{N\to\infty} \sqrt[N]{|A_N|} = 1 \end{cases}$$

2.6 Comparison Test

If $0 \leq A_N \leq B_N$, and $\displaystyle\sum_{N=0}^{\infty} A_N = \infty$, then $\displaystyle\sum_{N=0}^{\infty} B_N = \infty$.

If $0 \leq A_N \leq B_N$, and $\displaystyle\sum_{N=0}^{\infty} B_N < \infty$, then $\displaystyle\sum_{N=0}^{\infty} A_N < \infty$.

Example:
Is $\displaystyle\sum_{N=1}^{\infty} \frac{lnN}{N}$ convergent?
$0 < \frac{1}{N} < \frac{lnN}{N}$
And $\displaystyle\sum_{N=1}^{\infty} \frac{1}{N} = \infty$, so $\displaystyle\sum_{N=1}^{\infty} \frac{lnN}{N} = \infty$. (divergent)

2.7 Limit Comparison Test

If $A_N > 0$, $B_N > 0$, and $0 < \lim_{N\to\infty} \frac{A_N}{B_N} < \infty$, then $\displaystyle\sum_{N=0}^{\infty} A_N$ and $\displaystyle\sum_{N=0}^{\infty} B_N$ converge or diverge together.

Example:
Does $\displaystyle\sum_{N=0}^{\infty} \frac{N^3+2N}{3N^6+4}$ converge?
Compare with $\displaystyle\sum_{N=1}^{\infty} \frac{1}{N^3}$.

$$\lim_{N\to\infty} \frac{\frac{N^3+2N}{3N^6+4}}{\frac{1}{N^3}} = \lim_{N\to\infty} \frac{(N^3)(N^3+2N)}{3N^6+4} = \lim_{N\to\infty} \frac{N^6+2N^4}{3N^6+4} = \lim_{N\to\infty} \frac{N^6}{3N^6} = \frac{1}{3} \quad (4)$$

So since $\sum_{N=1}^{\infty} \frac{1}{N^3}$ converges by the P-test, $\sum_{N=0}^{\infty} \frac{N^3+2N}{3N^6+4}$ must converge as well.

3 Alternating Series

When $A_N \geq 0$, $\sum_{N=0}^{\infty} (-1)^N A_N$ is called an alternating series.

If A_N is monotonically decreasing and $A_N \Rightarrow 0$, then $\sum_{N=0}^{\infty} (-1)^N A_N$ is convergent.

Also, these series can be estimated.

If $\sum_{N=0}^{\infty} (-1)^N A_N = L$, then $\left| L - \sum_{N=0}^{M} (-1)^N A_N \right| \leq A_M$.

4 Finite Sums

$$\sum_{j=0}^{N} R^j = \frac{1-R^{N+1}}{1-R} \tag{5}$$

$$\sum_{j=1}^{N} 1 = N \tag{6}$$

$$\sum_{j=1}^{N} j = \frac{1}{2}N^2 + \frac{1}{2}N = \frac{N(N+1)}{2} \tag{7}$$

$$\sum_{j=1}^{N} j^2 = \frac{1}{3}N^3 + \frac{1}{2}N^2 + \frac{1}{6}N \tag{8}$$

$$\sum_{j=1}^{N} j^k \approx \frac{N^{k+1}}{k+1} \tag{9}$$