1 Functions vs. Vector Fields

A function, $f(x, y, z)$, takes a point (x,y,z) to a number.
For example:
$f(x, y, z) = x - y$
$f(x, y, z) = \sin(x + z^2) - 2$

A vector field $(f(x, y, z), g(x, y, z), h(x, y, z))$ takes a vector (or point) to another vector.
For example:
$(x, y + z, -x^2)$
(yz, xz, xy)
Examples of vector fields include: fluid/air flow, wind patterns, electric/magnetic/gravitational fields, rain, traffic flow, ...

The only vector field you'll be concerned with in this class is the "Gradient of F", which takes a point (x, y, z) and tells you the direction in which F increases the most from that point.

2 The Gradient $\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)$

1) ∇F points in the direction of greatest increase and $|\nabla F|$ is the rate of increase.
2) ∇F is perpendicular to the surface defined by $F = k$ (where k is a number) everywhere on that surface.
3) Maxima and Minima can occur when $\nabla F = \vec{0}$.

Q: $F(x, y, z) = x$. In which direction is this function increasing the fastest?
A: $\nabla F = (1, 0, 0)$

Q: Find a vector perpendicular to the surface $z = x^2 + y - 1$ at the point $(-1, 1, 1)$.
A: $F(x, y, z) = x^2 + y - 1 - z = 0$
$\nabla F(x, y, z) = (2x, 1, -1)$
$\nabla F(-1, 1, 1) = (-2, 1, -1)$ is perpendicular to $z = x^2 + y - 1$ at the point $(-1, 1, 1)$.

2.1 Directional Derivatives

"The rate at which $F(x, y, z)$ increases in the direction of $\hat{u}$" is written "$D_{\hat{u}}F$". In general "$\hat{u}$" (instead of "$\vec{u}$ ") is used because only the direction of $\hat{u}$ is important.

$$D_{\hat{u}}F = \nabla F \cdot \hat{u} = |\nabla F||\hat{u}| \cos(\theta) = |\nabla F| \cos(\theta) \tag{1}$$

Notice that $D_{\hat{u}}F$ is largest when $\theta = 0$, i.e. when moving in the direction of ∇F. You can also think of $D_{\hat{u}}F$ as the length of the projection of ∇F onto $\vec{u}$. In this way you don't have to worry about the "u hat" thing.

Q: At what rate is $F(x, y) = x^2 + 3y^2$ increasing at the point $\vec{P} = (-2, 1)$ and in the direction $\hat{u} = (0, 1)$.

A: $D_{\hat{u}}F = (2x, 6y) \cdot (0, 1) = 6y$

$D_{(0,1)}F(-2, 1) = 6(-2) = -12$. This means that F is in fact decreasing at the point (-2,1) in the direction (0,1).

2.2 Extremal Values

To find the critical points of a one variable function you'd take the derivative and set it equal to zero. To determine if the points you find are maxima or minima you can do the second derivative test.

To find the critical points of a two variable function you'd find the gradient, and set each term equal to zero. Instead of the second derivative test, however, you have this:

$$D = F_{xx}F_{yy} - (F_{xy})^2 \tag{2}$$

Plugging in a point you find a value for D.

$D > 0$	$\rightarrow$	the point is an extremum
$D < 0$	$\rightarrow$	the point is a saddle point
$D = 0$	$\rightarrow$	inconclusive

Q: Find the extreme values of $F(x, y) = y^3 - \frac{1}{2}y^2 + x^2$.

A: $\nabla F = (2x, 3y^2 - y) = (2x, y(3y - 1)) = \vec{0}$

$$\begin{cases} 2x = 0 \\ y(3y-1) = 0 \end{cases}$$

$$\begin{cases} x = 0 \\ y = 0 \text{ or } y = \frac{1}{3} \end{cases}$$

The two possible points to look at are: $(0, 0)$ and $(0, \frac{1}{3})$.

$F_x = 2x$

$F_y = 3y^2 - y$

$F_{xx} = 2$

$F_{yy} = 6y - 1$

$F_{xy} = 0$

$D = F_{xx}F_{yy} - (F_{xy})^2 = 12y - 2$

At the point $(0, 0)$, $D = -2 < 0$. So $(0, 0)$ is a saddle point.

At the point $(0, \frac{1}{3})$, $D = 4 > 0$. So $(0, \frac{1}{3})$ is either a maximum or minimum. To check take a look at either F_{xx} or F_{yy}. $F_{xx} = 2 > 0$ so $(0, \frac{1}{3})$ is a minimum.

3 Lagrange Multipliers

Lagrange Multipliers allow you to find extreme values of a function $F(x, y, z)$, given a restriction $G(x, y, z) = k$. The restriction $G(x, y, z) = k$ forms a surface, and (by the second property of gradients) $\nabla G(x, y, z)$ is perpendicular to the surface at every point.

If $\nabla F(x_0, y_0, z_0)$ is not perpendicular to the surface $G(x, y, z) = k$, then you can move along the surface in a direction $\hat{u}$ that increases the value of $F(x, y, z)$. $D_{\hat{u}}F(x_0, y_0, z_0) = \nabla F(x_0, y_0, z_0) \cdot \hat{u} > 0$.

If $\nabla F(x_0, y_0, z_0)$ is perpendicular to the surface $G(x, y, z) = k$, then there is no direction that will increase $F(x, y, z)$. $D_{\hat{u}}F(x_0, y_0, z_0) = \nabla F(x_0, y_0, z_0) \cdot \hat{u} = 0$, since every direction in the surface is perpendicular to $\nabla F(x_0, y_0, z_0)$.

Finally, if ∇G and ∇F are both perpendicular to the surface $G(x, y, z) = k$, then they must be pointing in the same direction. It helps to understand the reasoning, but if you can't, the most important thing is:

$$\nabla F = \lambda \nabla G \tag{3}$$

Q: What is the highest and lowest value of $F = 3x^2 + y^2 + z$ on the plane $x + y - z = 2$?
A:
$F = 3x^2 + y^2 + z$
$G = x + y - z = 2$
$\nabla F = (6x, 2y, 1)$
$\nabla G = (1, 1, -1)$

$$\begin{matrix} \nabla F = \lambda \nabla G \\ x + y - z = 2 \end{matrix} \rightarrow \begin{matrix} 6x = \lambda \\ 2y = \lambda \\ 1 = -\lambda \\ x + y - z = 2 \end{matrix} \rightarrow \begin{matrix} x = -\frac{1}{6} \\ y = -\frac{1}{2} \\ x + y - z = 2 \end{matrix} \rightarrow \begin{matrix} x = -\frac{1}{6} \\ y = -\frac{1}{2} \\ z = -\frac{8}{3} \end{matrix}$$

So $(-\frac{1}{6}, -\frac{1}{2}, -\frac{8}{3})$ is an extremum.
$F(-\frac{1}{6}, -\frac{1}{2}, -\frac{8}{3}) = 3(-\frac{1}{6})^2 + (-\frac{1}{2})^2 - \frac{8}{3} = \frac{3}{36} + \frac{1}{4} - \frac{8}{3} = \frac{1+3-32}{12} = -\frac{28}{32} = -\frac{7}{8}$
If you let x or y become large numbers, then $F(x, y, z)$ will become large. So $(-\frac{1}{6}, -\frac{1}{2}, -\frac{8}{3})$ is a minimum, and there is no maximum.

Q: Find all the extremal values of $F(x, y) = x(y^2 - 1)$ when $G(x, y) = x^2 + \frac{y^2}{4} \leq 1$.

A: First look at the interior ($x^2 + \frac{y^2}{4} < 1$). The max and min here can be found by setting $\nabla F = \vec{0}$.

$$\nabla F = (y^2 - 1, 2xy) = (0, 0) \rightarrow \begin{matrix} y^2 - 1 = 0 \\ 2xy = 0 \end{matrix} \rightarrow \begin{matrix} y = \pm 1 \\ \pm 2x = 0 \end{matrix} \rightarrow \begin{matrix} y = \pm 1 \\ x = 0 \end{matrix}$$

So the two critical points are: $(0,-1)$ and $(0,1)$.

$F_x = y^2 - 1$

$F_{xx} = 0$

$F_y = 2xy$

$F_{yy} = 2x$

$F_{xy} = 2y$

$D = F_{xx}F_{yy} - (F_{xy})^2 = -4y^2$

Plugging in either point you'll see that $D < 0$. So $(0,-1)$ and $(0,1)$ are saddle points. What remains is to look at the boundary. Here the goal is to find the extremum of $F(x,y) = x(y^2-1)$ restricted to $G(x,y) = x^2 + \frac{y^2}{4} = 1$. This requires Lagrange Multipliers.

$\nabla G = (2x, \frac{y}{2})$

$$\begin{matrix} \nabla F = \lambda \nabla G \\ x^2 + \frac{y^2}{4} = 1 \end{matrix} \rightarrow \begin{matrix} y^2 - 1 = \lambda 2x \\ 2xy = \lambda \frac{y}{2} \\ x^2 + \frac{y^2}{4} = 1 \end{matrix} \rightarrow \begin{matrix} y^2 - 1 = 8x^2 \\ 4x = \lambda \\ 8x^2 + 2y^2 = 8 \end{matrix} \rightarrow \begin{matrix} y^2 - 1 = 8x^2 \\ y^2 - 1 + 2y^2 = 8 \end{matrix} \rightarrow \begin{matrix} x^2 = \frac{y^2-1}{8} \\ y^2 = 3 \end{matrix}$$

$$\rightarrow \begin{matrix} x^2 = \frac{y^2-1}{8} \\ y = \pm\sqrt{3} \end{matrix} \rightarrow \begin{matrix} x^2 = \frac{2}{8} = \frac{1}{4} \\ y = \pm\sqrt{3} \end{matrix} \rightarrow \begin{matrix} x = \pm\frac{1}{2} \\ y = \pm\sqrt{3} \end{matrix}$$

So the critical points on the boundary are: $(\frac{1}{2}, \sqrt{3})$, $(\frac{1}{2}, -\sqrt{3})$, $(-\frac{1}{2}, \sqrt{3})$, and $(-\frac{1}{2}, -\sqrt{3})$.

$F(\frac{1}{2}, \pm\sqrt{3}) = \frac{1}{2}(3-1) = 1$

$F(-\frac{1}{2}, \pm\sqrt{3}) = -\frac{1}{2}(3-1) = -1$

The minimum value of F is -1, and the maximum value is 1.