Fundamental Limits of MIMO Capacity for Spatially Constrained Arrays

Tony S. Pollock, Thushara D. Abhayapala, and Rodney A. Kennedy

***Abstract*—In this paper we investigate the capacity behavior of spatial constrained multiple-antenna array communications. By increasing the number of antennas within a fixed region of space the antenna array becomes dense and spatial correlation inhibits capacity growth. Using a novel spatial channel model we show that the underlying physics of wave propagation limits the capacity of constrained arrays. A theoretically derived antenna saturation point is shown to exist for dense array MIMO systems, at which there is no capacity growth with increasing antenna numbers. We show this saturation point increases linearly with the radius of the region, and that it naturally lends itself to a definition for the theoretical maximum capacity for a fixed region of space.**

***Index Terms*— MIMO, wireless, capacity, diversity, multielement antenna arrays, dense arrays.**

I. Introduction

Multiple-Input Multiple-Output (MIMO) communications systems using multi-antenna arrays simultaneously during transmission and reception have generated significant interest in recent years. Theoretical work of [1] and [2] showed the potential for significant capacity increases in wireless channels via spatial multiplexing with sparse antenna arrays. However, in reality the capacity is significantly reduced when the antennas are placed close together so the signals received by different antennas become correlated, corresponding to a reduction of the effective number of sub-channels between transmit and receive antennas [3].

Previous studies have given insights and bounds into the effects of correlated channels [3–7], these can generally be grouped into two methods; eigenvalue decomposition approach [3–5] and correlation matrix approach [6, 7], however most have been for a limited set of channel realizations and/or antenna configurations.

In contrast, in this paper we approach the MIMO capacity problem from a physical wave field perspective. By using the underlying physics of free-space wave propagation we can explore the fundamental limits of capacity due to constraints imposed by the basic laws governing wave field behavior. In particular, using a modal expansion for free-space wave propagation we show that there exists a maximum achievable capacity for a fixed region of space. Furthermore, we theoretically analyze the effect on capacity of increasing numbers of antennas within this fixed region. As the number of antennas grows the antenna array becomes dense and spatial correlation significantly limits the capacity. Under these conditions we show a theoretical saturation point exists, where no further capacity gain is achieved with increasing numbers of antennas.

Recent independent works [8, 9] have studied the capacity of dense linear arrays, however, to the authors' knowledge no work exists for arbitrary array geometries, or has given a theoretical definition of the antenna saturation or maximum achievable capacity for a fixed region of space, as addressed here.

II. Ergodic Capacity of MIMO Systems

Consider a MIMO system consisting of n_T transmitters and n_R receivers, then the ergodic capacity is given by [1, 2],

$$\widetilde{C} = E\left\{\log\left|\boldsymbol{I}_{n_\mathrm{R}} + \frac{1}{\sigma^2}\boldsymbol{H}\boldsymbol{\Phi}_\mathrm{T}\boldsymbol{H}^\dagger\right|\right\} \tag{1}$$

where $\boldsymbol{H}$ is the $n_\mathrm{R} \times n_\mathrm{T}$ random flat fading channel matrix assumed known at the receiver, $\boldsymbol{\Phi}_\mathrm{T}$ is the transmitter covariance matrix, η is the average signal-to-noise ratio (SNR) at each receiver branch, $\boldsymbol{I}_{n_\mathrm{R}}$ is the $n_\mathrm{R} \times n_\mathrm{R}$ identity matrix, $|\cdot|$ is the determinant operator, and $\dagger$ is the complex conjugate transpose.

In the following we will assume that the channel matrix $\boldsymbol{H}$ is known only to the receiver, in this case it is optimal to let the transmitted signals be statistically independent equal power components each with a Gaussian distribution [1], then $\boldsymbol{\Phi}_\mathrm{T} = (P_\mathrm{tx}/n_\mathrm{T})\boldsymbol{I}_{n_\mathrm{T}}$, where P_tx is the total transmitted power (independent of the number of transmitters). Hence, for normalized channel gains, $E\left\{|h_{rt}|^2\right\} = 1$, where h_{rt} is the channel gain from tth transmitter to the rth receiver, we have

$$\widetilde{C} = E\left\{\log\left|\boldsymbol{I}_{n_\mathrm{R}} + \frac{\eta}{n_\mathrm{R} n_\mathrm{T}}\boldsymbol{H}\boldsymbol{H}^\dagger\right|\right\} \tag{2}$$

where we have introduced the scaling factor $1/n_\mathrm{R}$ which ensures the total received power remains independent of the number of receiver antennas [10], thereby allowing us to study the capacity growth limits due to modal saturation effects independent of antenna array gains. Strictly speaking the capacity expression (2) should be referred to as the *normalized capacity* (in the array-gain sense), however, in the following we will simply use the term capacity and clearly state when we mean otherwise.

Equation (2) is often used in Monte Carlo simulations to provide capacity results for various MIMO systems. However, current channel matrix models do not include spatial information (antenna locations) explicitly, although it is represented by the correlation between channel matrix elements it has no direct realizable physical representation and therefore does not easily

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lend itself to insightful capacity results. In particular, of interest is the effect on channel capacity of antenna placement at both the transmitter and the receiver, particularly in the realistic case when antenna arrays are restricted in size, along with non-isotropic scattering environments. Therefore, with additional theory for modelling scattering environments which we refine here, we derive a model which can be readily reconciled with a multitude of scattering models and antenna configurations and allows us to uncover valuable insights into MIMO capacity.

III. Channel Model

Consider a MIMO system with $n_{\rm T}$ transmit antennas located at positions $\boldsymbol{x}_t$, $t = 1, 2, \ldots, n_{\rm T}$ relative to the transmitter array origin, and $n_{\rm R}$ receiver antennas located at positions $\boldsymbol{y}_r$, $r = 1, 2, \ldots, n_{\rm R}$, relative to the receiver array origin. Denote

$$r_{\rm T} = \max_{t=1,2,\ldots,n_{\rm T}} \|\boldsymbol{x}_t\| \tag{3}$$

and

$$r_{\rm R} = \max_{r=1,2,\ldots,n_{\rm R}} \|\boldsymbol{y}_r\| \tag{4}$$

as the radius of the spheres that contain all the transmitter and receiver antennas within, respectively, as shown in Fig. 1. We also assume that the scatterers are distributed in the farfield from both the transmitter and receiver sensors, therefore we define scatter-free transmitter and receiver balls of radius $r_{\rm TS}$ ($> r_{\rm T}$) and $r_{\rm RS}$ ($> r_{\rm R}$), respectively[1].

Let $\boldsymbol{x} = [x_1, x_2, \ldots, x_{n_{\rm T}}]^T$ be the vector of symbols sent by the $n_{\rm T}$ transmitting antennas, where $[\cdot]^T$ denotes the vector transpose, then the signal leaving the scatter-free transmitter ball along direction $\widehat{\boldsymbol{\phi}}$ is given by

$$\Phi(\widehat{\boldsymbol{\phi}}) = \sum_{t=1}^{n_{\rm T}} x_t e^{ik\boldsymbol{x}_t\cdot\widehat{\boldsymbol{\phi}}} \tag{5}$$

where $k = 2\pi/\lambda$ is the wavenumber with λ the wavelength, and $\widehat{\boldsymbol{\phi}}$ is a unit vector pointing away from the origin of the transmitter scatter-free ball. Denote $g(\widehat{\boldsymbol{\phi}}, \widehat{\boldsymbol{\psi}})$ as the complex gain of the scattering environment for a signal leaving the transmitter scatter-free ball at an angle $\widehat{\boldsymbol{\phi}}$, and entering the receiver scatter-free ball at an angle $\widehat{\boldsymbol{\psi}}$, then

$$\Psi(\widehat{\boldsymbol{\psi}}) = \int_{\mathbb{S}^2} \Phi(\widehat{\boldsymbol{\phi}}) g(\widehat{\boldsymbol{\phi}}, \widehat{\boldsymbol{\psi}}) ds(\widehat{\boldsymbol{\phi}}) \tag{6}$$

represents the signal entering the scatter-free receiver ball from direction $\widehat{\boldsymbol{\psi}}$, where $ds(\widehat{\boldsymbol{\phi}})$ is a surface element of the unit sphere $\mathbb{S}^2$ with unit normal $\widehat{\boldsymbol{\phi}}$. Finally, the received signal y_r at position $\boldsymbol{y}_r$ is given by

$$y_r = \int_{\mathbb{S}^2} \Psi(\widehat{\boldsymbol{\psi}}) e^{-ik\boldsymbol{y}_r\cdot\widehat{\boldsymbol{\psi}}} ds(\widehat{\boldsymbol{\psi}}) + w_r \tag{7}$$

where w_r is the additive white Gaussian noise at the rth receiver.

[1]The Rayleigh distance [11] gives the distance for farfield approximation from the array origin as $d = 2\ell^2/\lambda$, where ℓ is the largest array dimension and λ is the wavelength.

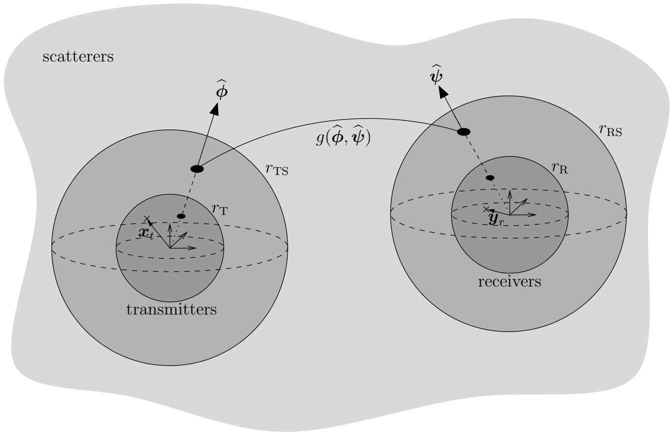

Fig. 1: Scattering model for a flat fading MIMO system. $r_{\rm T}$ and $r_{\rm R}$ are the radii of spheres which enclose the transmitter and the receiver arrays. $g(\widehat{\boldsymbol{\phi}}, \widehat{\boldsymbol{\psi}})$ represents the gain of the complex scattering environment for signals leaving the transmitter scattering free region from direction $\widehat{\boldsymbol{\phi}}$ and arriving at the receiver scattering free region from direction $\widehat{\boldsymbol{\psi}}$.

Let $\boldsymbol{y} = [y_1, y_2, \ldots, y_{n_{\rm R}}]^T$ and $\boldsymbol{w} = [w_1, w_2, \ldots, w_{n_{\rm R}}]^T$, then we can write (7) in vector form

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{w} \tag{8}$$

where $\boldsymbol{H}$ represents the $n_{\rm R} \times n_{\rm T}$ channel matrix, with (r, t)th element

$$\boldsymbol{H}|_{rt} = \int_{\mathbb{S}^2} g(\widehat{\boldsymbol{\phi}}, \widehat{\boldsymbol{\psi}}) e^{ik\boldsymbol{x}_t\cdot\widehat{\boldsymbol{\phi}}} e^{-ik\boldsymbol{y}_r\cdot\widehat{\boldsymbol{\psi}}} ds(\widehat{\boldsymbol{\phi}}) ds(\widehat{\boldsymbol{\psi}}) \tag{9}$$

representing the channel gain between the tth transmit antenna and the rth receive antenna.

Note that to ensure the transmit antenna to receiver antenna channel gains (9) are normalized we require the scattering region gains $g(\widehat{\boldsymbol{\phi}}, \widehat{\boldsymbol{\psi}})$ normalized such that

$$\int_{\mathbb{S}^2} E\left\{|g(\widehat{\boldsymbol{\phi}}, \widehat{\boldsymbol{\psi}})|^2\right\} ds(\widehat{\boldsymbol{\phi}}) ds(\widehat{\boldsymbol{\varphi}}) = 1 \tag{10}$$

Although the channel model (9) relates well with a physically realizable system it is however difficult to evaluate or simulate due its integral representation. In the following section we exploit the underlying physics of wave propagation to reduce $\boldsymbol{H}$ to an easily computable form, which will allow us to explore its properties further.

A. Channel Matrix Decomposition

Consider a 2D scattering environment where the signals are height invariant and arrive only from the azimuthal plane[2]. In this case we can use the Jacobi-Anger 2D modal expansion of plane waves [12]

$$e^{ik\boldsymbol{x}\cdot\widehat{\boldsymbol{\phi}}} = \sum_{n=-\infty}^{\infty} \overline{\mathcal{J}_n(\boldsymbol{x})} e^{in\phi} \tag{11}$$

[2]Similar results can be obtained for 3D environment, however we omit these here for the sake of brevity.

where $\overline{f(\boldsymbol{x})}$ is the complex conjugate of $f(\boldsymbol{x})$, and define

$$\mathcal{J}_n(\boldsymbol{x}) \triangleq J_n(k\|\boldsymbol{x}\|)e^{in(\phi_x - \pi/2)} \tag{12}$$

with $\boldsymbol{x} = (\|\boldsymbol{x}\|, \phi_x)$, $\widehat{\boldsymbol{\phi}} = (1, \phi)$, and $J_n(\cdot)$ are the Bessel functions of the first kind. Bessel functions $J_n(z)$, $|n| > 0$ exhibit spatially high pass behavior, that is, for fixed order n, $J_n(z)$ starts small and becomes significant for arguments $z \approx \mathcal{O}(n)$. Therefore, for a fixed argument z, the Bessel function $J_n(z) \approx 0$ for all but a finite set of low order $n \leq N$, hence (11) is well approximated by the finite sum

$$e^{ik\boldsymbol{x}\cdot\widehat{\boldsymbol{\phi}}} = \sum_{n=-N}^{N} \overline{\mathcal{J}_n(\boldsymbol{x})}e^{in\phi}. \tag{13}$$

In [13] it was shown that $J_n(z) \approx 0$ for $n > \lceil ze/2 \rceil$, with $\lceil \cdot \rceil$ the ceiling operator, we can then define

$$N_{\mathrm{T}} \triangleq \lceil \pi e r_{\mathrm{T}}/\lambda \rceil \tag{14}$$
$$N_{\mathrm{R}} \triangleq \lceil \pi e r_{\mathrm{R}}/\lambda \rceil \tag{15}$$

such that the expansions

$$e^{ik\boldsymbol{x}_t\cdot\widehat{\boldsymbol{\phi}}} = \sum_{n=-N_{\mathrm{T}}}^{N_{\mathrm{T}}} \overline{\mathcal{J}_n(\boldsymbol{x}_t)}e^{in\phi} \tag{16}$$

$$e^{-ik\boldsymbol{y}_r\cdot\widehat{\boldsymbol{\psi}}} = \sum_{m=-N_{\mathrm{R}}}^{N_{\mathrm{R}}} \mathcal{J}_m(\boldsymbol{y}_r)e^{-im\psi} \tag{17}$$

hold for every antenna within the transmitter and receiver circular regions of radii r_{T} and r_{R}, respectively.

Substitution of (16) and (17) into (9) gives the channel matrix decomposition

$$\boldsymbol{H} = \boldsymbol{J}_{\mathrm{R}}\boldsymbol{H}_0\boldsymbol{J}_{\mathrm{T}}^{\dagger} \tag{18}$$

where $\boldsymbol{J}_{\mathrm{T}}$ is the $n_{\mathrm{T}} \times (2N_{\mathrm{T}}+1)$ transmitter scatter-free region matrix,

$$\boldsymbol{J}_{\mathrm{T}} = \begin{bmatrix} \mathcal{J}_{-N_{\mathrm{T}}}(\boldsymbol{x}_1) & \cdots & \mathcal{J}_{N_{\mathrm{T}}}(\boldsymbol{x}_1) \\ \vdots & \ddots & \vdots \\ \mathcal{J}_{-N_{\mathrm{T}}}(\boldsymbol{x}_{n_{\mathrm{T}}}) & \cdots & \mathcal{J}_{N_{\mathrm{T}}}(\boldsymbol{x}_{n_{\mathrm{T}}}) \end{bmatrix}, \tag{19}$$

$\boldsymbol{J}_{\mathrm{R}}$ is the $n_{\mathrm{R}} \times (2N_{\mathrm{R}}+1)$ receiver scatter-free region matrix,

$$\boldsymbol{J}_{\mathrm{R}} = \begin{bmatrix} \mathcal{J}_{-N_{\mathrm{R}}}(\boldsymbol{y}_1) & \cdots & \mathcal{J}_{N_{\mathrm{R}}}(\boldsymbol{y}_1) \\ \vdots & \ddots & \vdots \\ \mathcal{J}_{-N_{\mathrm{R}}}(\boldsymbol{y}_{n_{\mathrm{R}}}) & \cdots & \mathcal{J}_{N_{\mathrm{R}}}(\boldsymbol{y}_{n_{\mathrm{R}}}) \end{bmatrix}, \tag{20}$$

and $\boldsymbol{H}_0$ is a $(2N_{\mathrm{R}}+1) \times (2N_{\mathrm{T}}+1)$ scattering channel matrix. For some $p \in [1, 2N_{\mathrm{R}}+1]$ and $q \in [1, 2N_{\mathrm{T}}+1]$ let $n = q - N_{\mathrm{T}} - 1$ and $m = p - N_{\mathrm{R}} - 1$, then the scattering-region channel matrix $\boldsymbol{H}_0$ has elements

$$\boldsymbol{H}_0|_{pq} = \int_0^{2\pi} g(\phi,\psi)e^{in\phi}e^{-im\psi}d\phi d\psi \tag{21}$$

representing the complex gain of the scattering channel between the qth mode of the scatter-free transmitter region and the pth mode of the scatter-free receiver region.

We may simplify (21) by further observing that $h_{0,pq}$ are the complex coefficients β_m^n of the 2D Fourier series for the periodic scattering gain function $g(\phi,\psi)$,

$$g(\phi,\psi) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \beta_m^n e^{-in\phi}e^{im\psi} \tag{22}$$

therefore

$$\boldsymbol{H}_0|_{pq} = \beta_m^n = \beta_{p-N_{\mathrm{R}}-1}^{q-N_{\mathrm{T}}-1}. \tag{23}$$

Thus we can parameterize the random scattering channel by the complex random coefficients β_m^n, $m \in [-N_{\mathrm{R}}, N_{\mathrm{R}}]$, $n \in [-N_{\mathrm{T}}, N_{\mathrm{T}}]$, giving

$$\boldsymbol{H}_0 = \begin{bmatrix} \beta_{-N_{\mathrm{R}}}^{-N_{\mathrm{T}}} & \cdots & \beta_{-N_{\mathrm{R}}}^{N_{\mathrm{T}}} \\ \vdots & \ddots & \vdots \\ \beta_{N_{\mathrm{R}}}^{-N_{\mathrm{T}}} & \cdots & \beta_{N_{\mathrm{R}}}^{N_{\mathrm{T}}} \end{bmatrix} \tag{24}$$

The channel matrix decomposition (18) separates the channel into three distinct regions of signal propagation: transmitters, scatterers, and receivers, as shown in Fig. 1. Both the scatter-free transmitter $\boldsymbol{J}_{\mathrm{T}}$ and receiver $\boldsymbol{J}_{\mathrm{R}}$ matrices, describing the array geometry, are constant for fixed antenna locations within the spatial region. Conversely, for a random scattering environment the scattering channel matrix $\boldsymbol{H}_0$ will have random elements, however, unlike other channel models, which don't explicitly use the antenna positions (or are constrained to particular scattering environments or arrays) we can now simulate the channel for various array geometries and scattering environments.

IV. Antenna Saturation of Spatially Constrained Arrays

It is well known that the rank of the channel matrix $\boldsymbol{H}$ gives the effective number of independent parallel channels between the transmit and receive antenna arrays, and thus determines the capacity of the system. For the decomposition (18) we have $\text{rank}(\boldsymbol{H}) = \min\{\text{rank}(\boldsymbol{J_T}), \text{rank}(\boldsymbol{J_R}), \text{rank}(\boldsymbol{H_0})\}$, which, for a large number of antennas in a rich scattering environment, becomes $\min\{2N_{\mathrm{T}}+1, 2N_{\mathrm{R}}+1\}$. Therefore we see that the number of modes for the scatter-free transmit and receive regions limit the capacity of the system, this key result provides the motivation for what follows next.

A. Convergence of Ergodic Capacity

Let $\widetilde{\boldsymbol{H}} = \boldsymbol{H}_0\boldsymbol{J}_{\mathrm{T}}^{\dagger} = [\widetilde{\boldsymbol{h}}_1, \widetilde{\boldsymbol{h}}_2, \ldots, \widetilde{\boldsymbol{h}}_{n_{\mathrm{T}}}]$, where $\widetilde{\boldsymbol{h}}_t$ is the $(2N_{\mathrm{R}}+1) \times 1$ vector corresponding to the complex channel gains from the tth transmitter to the $2N_{\mathrm{R}}+1$ receiver scatter-free region modes, then the receiver modal correlation matrix is defined as

$$\boldsymbol{R}_{\widetilde{\boldsymbol{H}}} \triangleq E\{\widetilde{\boldsymbol{h}}_t\widetilde{\boldsymbol{h}}_t^{\dagger}\}, \ \forall t \tag{25}$$

where element $\boldsymbol{R}_{\widetilde{\boldsymbol{H}}}|_{mm'}$ is the modal correlation between two modes m and m' associated with the receiver scatter-free region.

Consider the situation where the transmitter array of radius r_{T} has optimally placed (uncorrelated) $n_{\mathrm{T}} = 2N_{\mathrm{T}} + 1$ antennas, corresponding to independent $\widetilde{\boldsymbol{h}}_t$ vectors, then the sample receiver modal correlation matrix is given by

$$\widehat{\boldsymbol{R}}_{\widetilde{\boldsymbol{H}}} = \frac{1}{n_{\mathrm{T}}} \sum_{t=1}^{n_{\mathrm{T}}} \widetilde{\boldsymbol{h}}_t \widetilde{\boldsymbol{h}}_t^{\dagger} \tag{26}$$

which converges to $\boldsymbol{R}_{\widetilde{\boldsymbol{H}}}$ for large numbers of antennas ($r_T \to \infty$). Observing that $\widetilde{\boldsymbol{H}}\widetilde{\boldsymbol{H}}^{\dagger} = \sum_{t=1}^{n_{\mathrm{T}}} \widetilde{\boldsymbol{h}}_t \widetilde{\boldsymbol{h}}_t^{\dagger}$, then for a large number of uncorrelated transmit antennas the ergodic capacity (2) converges to the deterministic quantity C,

$$\lim_{r_{\mathrm{T}} \to \infty} \widetilde{C} = C \triangleq \log \left| \boldsymbol{I}_{n_{\mathrm{R}}} + \frac{\eta}{n_{\mathrm{R}}} \boldsymbol{J}_{\mathrm{R}} \boldsymbol{R}_{\widetilde{\boldsymbol{H}}} \boldsymbol{J}_{\mathrm{R}}^{\dagger} \right|. \tag{27}$$

This analytical capacity expression allows us to explore the effects on ergodic capacity of the scattering environment and receiver array geometry without lengthy Monte-Carlo simulations, as were previously required. By assuming a large transmitter region we have effectively removed any transmitter effect on capacity (since $2N_T + 1 \gg 2N_R + 1$), allowing us to independently analyze the effect on capacity due to constraining the receiver array.

B. Antenna Saturation Point

To ensure we only see the effects on capacity of spatially constraining the array we will assume the scatterers generate an isotropic diffuse field at the receiver (often referred to as a *rich* scattering environment) corresponding to independent elements β_m^n of the scattering channel matrix. Therefore, for a rich scattering environment, we have the maximum number of independent modes at the receiver and the receiver modal correlation matrix becomes $\boldsymbol{R}_{\widetilde{\boldsymbol{H}}} = \boldsymbol{I}_{2N_{\mathrm{R}}+1}$ and (27) reduces to

$$C = \log \left| \boldsymbol{I}_{n_{\mathrm{R}}} + \frac{\eta}{n_{\mathrm{R}}} \boldsymbol{J}_{\mathrm{R}} \boldsymbol{J}_{\mathrm{R}}^{\dagger} \right| \tag{28}$$

where the (q, r)th element of the receiver scatter-free matrix product $\boldsymbol{J}_{\mathrm{R}} \boldsymbol{J}_{\mathrm{R}}^{\dagger}$ is given by

$$\begin{aligned} \boldsymbol{J}_{\mathrm{R}} \boldsymbol{J}_{\mathrm{R}}^{\dagger}|_{qr} &= \sum_{n=-N_{\mathrm{R}}}^{N_{\mathrm{R}}} \mathcal{J}_n(\boldsymbol{y}_q) \overline{\mathcal{J}_n(\boldsymbol{y}_r)} \\ &= J_0(k \|\boldsymbol{y}_q - \boldsymbol{y}_r\|) \end{aligned} \tag{29}$$

which follows from a special case of Gegenbauer's Addition Theorem [14, pp. 363]. For a rich scattering environment $J_0(k\|\boldsymbol{y}_q - \boldsymbol{y}_r\|)$ gives the spatial correlation between the complex envelopes of the received signals at antennas q and r [15]. Therefore the capacity (28) will be maximized for the maximum number of antennas that can be placed within the region such that the spatial correlation between these antennas is zero.

Assuming we can place all n_{R} antennas such that the spatial correlation is zero between all the antennas, then the maximum capacity is given by

$$C_{\max}(n_{\mathrm{R}}) = n_{\mathrm{R}} \log(1 + \eta / n_{\mathrm{R}}) \tag{30}$$

however, as we will show next, if $n_{\mathrm{R}} \geq \mathrm{rank}(\boldsymbol{J}_{\mathrm{R}})$ then the capacity no longer grows with antenna number.

Let μ_ℓ, $\ell = 1, 2, \ldots, n_{\mathrm{R}}$ denote the singular values of the receiver scatter-free matrix $\boldsymbol{J}_{\mathrm{R}}$, ordered such that $\mu_\ell \geq \mu_{\ell+1}$, then we can express the capacity (28) as

$$C = \sum_{\ell} \log \left(1 + \eta \frac{\mu_\ell^2}{n_{\mathrm{R}}} \right). \tag{31}$$

Observing that $\sum_\ell \mu_\ell^2 = \mathrm{tr}(\boldsymbol{J}_{\mathrm{R}} \boldsymbol{J}_{\mathrm{R}}^{\dagger}) = n_{\mathrm{R}}$, where $\mathrm{tr}(\cdot)$ is the trace of the matrix, then denoting $\mu_{\ell, n_{\mathrm{R}1}}$ and $\mu_{\ell, n_{\mathrm{R}2}}$ as the singular values of $\boldsymbol{J}_{\mathrm{R}}$ for arrays within radius r_{R} with numbers of antennas $n_{\mathrm{R}1}, n_{\mathrm{R}2} \geq 2N_{\mathrm{R}} + 1$ respectively, we then have

$$\frac{1}{n_{\mathrm{R}1}} \sum_{\ell=1}^{n_{\mathrm{R}1}} \mu_{\ell, n_{\mathrm{R}1}}^2 = \frac{1}{n_{\mathrm{R}2}} \sum_{\ell=1}^{n_{\mathrm{R}2}} \mu_{\ell, n_{\mathrm{R}2}}^2. \tag{32}$$

By definition, there are only $\mathrm{rank}(\boldsymbol{J}_{\mathrm{R}})$ non-zero singular values (corresponding to the $2N_{\mathrm{R}} + 1$ independent receiver modes), therefore, assuming we can place $\mathrm{rank}(\boldsymbol{J}_{\mathrm{R}})$ antennas such that spatial correlation between each antenna is zero, giving constant and equal non-zero singular values, we have

$$\frac{\mu_{\ell,(n_{\mathrm{R}1})}^2}{n_{\mathrm{R}1}} = \frac{\mu_{\ell,(n_{\mathrm{R}2})}^2}{n_{\mathrm{R}2}}, \quad \ell \in [1, 2N_{\mathrm{R}} + 1]. \tag{33}$$

Therefore, letting $n_{\mathrm{R}1} = n_{\mathrm{sat}} \triangleq 2N_{\mathrm{R}} + 1$, and $n_{\mathrm{R}2} = n_{\mathrm{R}} \geq 2N_{\mathrm{R}} + 1$, (31) becomes

$$C_{\mathrm{sat}} \triangleq \sum_{\ell=1}^{n_{\mathrm{sat}}} \log \left(1 + \eta \frac{\mu_{\ell, n_{\mathrm{sat}}}^2}{n_{\mathrm{sat}}} \right), \quad n_{\mathrm{R}} \geq 2N_{\mathrm{R}} + 1 \tag{34}$$

which is independent of n_{R}, hence the capacity growth becomes zero once the number of antennas within the region reaches the saturation point given by $n_{\mathrm{sat}} = 2N_{\mathrm{R}} + 1$.

Combining (30) and (34) we get the theoretical maximum capacity for n_{R} antennas within a region of radius r_R as

$$C_{\max}(n_{\mathrm{R}}) = \begin{cases} n_{\mathrm{R}} \log(1 + \eta / n_{\mathrm{R}}) & \text{if } n_{\mathrm{R}} < n_{\mathrm{sat}} \\ C_{\mathrm{sat}} & \text{if } n_{\mathrm{R}} \geq n_{\mathrm{sat}} \end{cases} \tag{35}$$

V. Maximum Capacity for a Given Region of Space

From (35) we see the capacity for the region of radius r_{R} can be maximized using a minimum of $n_{\mathrm{R}} = n_{\mathrm{sat}} = 2N_{\mathrm{R}} + 1$ antennas placed optimally such that they are all uncorrelated, in this case all of the singular values of the receiver scatter-free matrix $\boldsymbol{J}_{\mathrm{R}}$ are non-zero and are all unity. Therefore we define the maximum capacity for the region as

$$C_{\max}(r_{\mathrm{R}}) = n_{\mathrm{sat}} \log(1 + \eta / n_{\mathrm{sat}}) \tag{36}$$

which is identical to the identity channel case ($\boldsymbol{H} = \boldsymbol{I}_{n_{\mathrm{sat}}}$) shown in [2], therefore for large n_{sat} (i.e., $r_{\mathrm{R}} \to \infty$) we have

$$\lim_{r_{\mathrm{R}} \to \infty} C_{\max}(r_{\mathrm{R}}) = C_{\mathrm{limit}} \triangleq \frac{\eta}{\ln 2} \tag{37}$$

which is the absolute maximum capacity if we allow the array to be unconstrained in size.

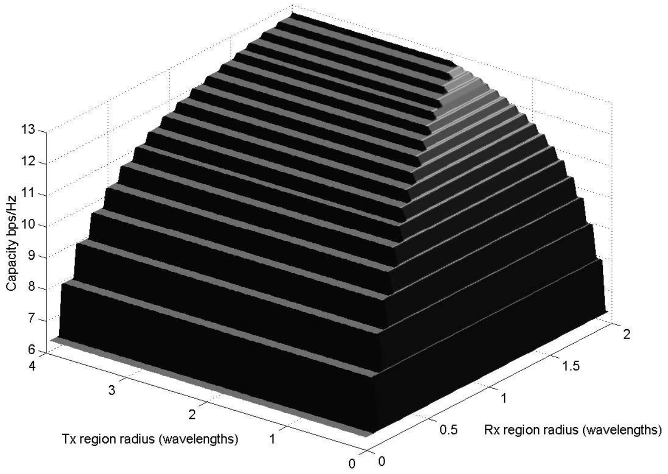

Fig. 2: Theoretical maximum MIMO Capacity of various sized transmitter (Tx) and receiver (Rx) regions in a rich scattering environment. A step in the capacity indicates where the radius becomes large enough to accommodate another mode in that region.

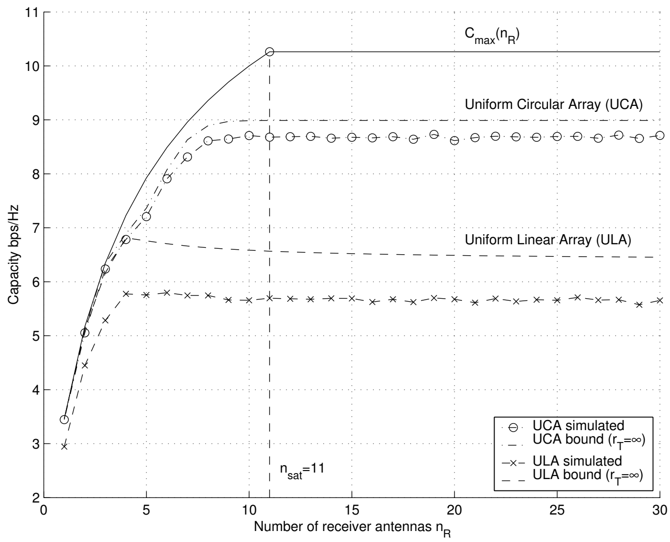

Fig. 3: Simulated capacity for uniform circular and uniform linear arrays in a rich scattering environment with transmitter radius $r_T = 5\lambda$ and receiver radius $r_R = 0.5\lambda$ for an increasing number of receiver antennas. Shown also is the theoretical maximum capacity (35) for the receiver region $C_{\max}(n_R)$ along with the upper bounds on capacity (28) when the number of transmitter antennas is large ($r_T = \infty$).

Since the MIMO capacity (2) is invariant with direction of information transmission [16], the previous sections hold for either the transmitter or receiver arrays, therefore we can generalize the result to define a maximum spatial capacity for any given scatter-free spatial region:

Definition 1 (Spatial Capacity Limit). *For a scatter-free circular region of radius r define the maximum achievable capacity for that region as*

$$C_{max}(r) \triangleq N_r \log\left(1 + \frac{\eta}{N_r}\right) \tag{38}$$

where

$$N_r = 2\lceil \pi e r/\lambda \rceil + 1 \tag{39}$$

with an absolute maximum capacity for regions of infinite radius given by (37).

Therefore we see that each scatter-free spatial region has its own maximum achievable capacity which depends only on the radius of the region, in particular we can define the maximum capacity for each end of a MIMO communications link and, provided we have a rich scattering environment, the maximum achievable capacity of the system will be the minimum of the two regions.

Fig. 2 shows the theoretical maximum capacity for a MIMO system in a rich scattering environment with transmitter and receiver regions of various radii. The modal nature of the system appears as steps in the capacity for increasing radii, that is, as we can only have integer numbers of modes a jump in capacity occurs where the radius of the region increases enough to accommodate another mode. Obviously if the antennas are non-optimally located, or the scattering environment is not rich, then the capacity may be a long way from this theoretical maximum, as we see in the following simulations.

VI. Numerical Results

We now present Monte-Carlo simulations to support the theoretical work of the previous sections. We consider the effect of increasing the number of receiver antennas n_{R} constrained within a scatter-free circular region of radius $r_{\mathrm{R}} = 0.5\lambda$, for a fixed number of transmitters constrained within a scatter-free circular region of radius $r_{\mathrm{T}} = 5\lambda$ ($n_{\mathrm{T}} = 2N_T + 1 = 87$) for SNR of 10dB. Fig. 3 shows the simulated capacity for Uniform Circular Arrays (UCA) and Uniform Linear Arrays (ULA) using the channel model presented in Section III-A and assuming a rich scattering environment (the elements of $\boldsymbol{H}_0$ are uncorrelated (i.i.d) random gaussian variables). Also shown is the theoretical maximum capacity (35) for the receiver region (given the large region size of the transmitter, the receiver region will determine the maximum achievable capacity of the system), along with the upper bounds (28) on capacity when the number of transmitter antennas is large ($r_T = \infty$).

As indicated by the vertical dashed line the antenna saturation point for the region (39) clearly shows where further increasing antenna numbers gives no further capacity gain. As expected, both the UCA and ULA do not optimally place the antennas for the given region, and as such we see the capacity is lower than the theoretical maximum capacity. These array geometries do not utilize the full set of independent transmission modes between regions, and therefore exhibit the behavior of optimally utilized regions of smaller radii, as shown by the lower antenna saturation points and capacity.

A measure of expected performance of a spatially constrained array can be obtained by considering the singular values of the scatter-free region matrix $\boldsymbol{J}_{\mathrm{R}}$ for each array geometry. For example, the condition numbers for the saturated UCA and ULA in a region of radius 0.5λ are approximately 9 and 280 respectively (compared to a condition number of 1 for optimally placed antennas). We are currently investigating and quantifying this performance measure, and will publish the results shortly.

VII. Discussion

We have derived a theoretical maximum capacity for a fixed region of space which depends only on the radius of the region. We also presented an antenna saturation point at which this maximum occurs, whereby further increases in the number of antennas fails to give further capacity gains. These results have significant implications for realizable MIMO systems, simply cramming more and more antennas into a region of space does not increase the capacity (it may however improve the diversity performance). Therefore we see that space also needs to be considered for the information bearing capacity of the system, along with the usual time-frequency constraint. However, these results show one cannot assign a spatial unit of capacity, as with the traditional time and frequency concepts, e.g., bps/Hz/m^3, simply because unlike the time and frequency case, doubling the region area or volume does not give a corresponding doubling of capacity.

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