Describing the Solution Set of a Linear System

The simplest system is one consisting of just one single *linear equation* like

$$3x+5y=7.$$

In a linear equation there are no squares or other powers of the variables, no fancy functions like sin(x) or e^x ; each variable has only a numerical coefficient. This linear equation can also be written as a matrix equation:

$$\begin{pmatrix} 3 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 7.$$

EXERCISE

The first vector is written as a row and the second as a column. Why is this? Would it be *wrong* to write the first vector as a column and the second as a row?

The product of a 1×2 times a 2×1 matrix is a 1×1 matrix (a number, in this case 3x+5y). The product of a 2×1 times a 1×2 would be a 2×2. Yes, it would be very wrong.

One solution to the linear equation 3x+5y=7 is (x,y)= (-1,2), but this is not the only one. There are *infinitely many solutions.* In fact, we can choose any value whatsoever for y and a corresponding value for x, namely x=(7-5y)/3, can always be found. The complete set of solutions can be described like this

$$\text{solution set} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x = \frac{7}{3} - \frac{5}{3} y,\ y \in \mathbf{R} \right\}$$

(which you would read as "the set of all the pairs (x,y) such that $x = \frac{7}{3} - \frac{5}{3} y$ **and y is any real number"). This is called a *parametric representation* of the solution set, and y is the *parameter.***

EXERCISE

How would you write the solution set if you wanted x to be the parameter?

like this:

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} : y = \frac{7}{5} - \frac{3}{5} x,\ x \in \mathbf{R} \right\}$$

A system of one or more linear equations, such as

$$\begin{aligned} 3x + 5y &= 7 \\ x + 2y &= 4 \\ 2x + 7y &= 23 \end{aligned}$$

can always be written as a single matrix equation:

$$\begin{pmatrix} 3 & 5 \\ 1 & 2 \\ 2 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 23 \end{pmatrix}$$

For this system, (x,y) = (-6,5) is the *unique solution.*

EXERCISE

Write the system

$$\begin{aligned} x &= 11 \\ y &= 13 \\ z-w &= 9 \end{aligned}$$

as a single matrix equation

answer:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 11 \\ 13 \\ 9 \end{pmatrix}$$

For some systems, *no solution exists*. Here are four examples of such systems:

$$\begin{aligned} x + y &= 3 \\ x + y &= 4 \end{aligned}$$

two equations,
two variables

$$\begin{aligned} x &= 3 \\ 3x &= 8 \end{aligned}$$

two equations,
one variable

$$\begin{aligned} 3x + 5y &= 8 \\ x + y &= 1 \\ 5x + 7y &= 9 \end{aligned}$$

three equations,
two variables

$$0x = 1$$

one equation,
one variable

EXERCISE

Can you convince yourself that none of these systems has a solution?

To see that the third system has no solution, notice that its first two equations imply that

$$3x + 7y = 10$$

Every system of linear equations falls into one of three possible categories. Either

there is a unique solution

or

there are infinitely many solutions

or

no solution exists.

These are the only possibilities. **Never in your lifetime nor in anyone else's will there be seen, for example, a system of linear equations having exactly two solutions.**

EXERCISE

To which category does this system belong?

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

x must equal 1, y must equal 2, and z must equal 3. But z can have any real value, so there are an infinite number of solutions.

Some Systems that are *Very* Easy to Solve

Consider this system of linear equations:

$$\begin{aligned} x_1 + 3x_2 \quad\quad - 2x_4 \quad &= 4 \\ x_3 - 5x_4 \quad &= 7 \\ x_5 &= 12 \end{aligned}$$

No work is required to solve this system because, in a sense, it is already solved the moment we write it down...

x_5 must equal 12. If we choose any values whatsoever for x_2 and x_4, the values of x_1 and x_3, are determined by the first two equations. This allows us to completely describe the set of solutions:

$$\{(x_1,x_2,x_3,x_4,x_5) : x_5=12,\ x_3=7+5x_4,\ x_1=4-3x_2+2x_4,\ x_2,x_4 \in R\}$$

in terms of two parameters x_2 and x_4. There are an infinite number of solutions – exactly one solution corresponds to each choice of values for x_2 and x_4.

EXERCISE

Can you describe what it is about this system that makes it so very easy to solve?

answer on next page

Answer: The previous system was so easy to solve because its augmented matrix

$$\left(\begin{array}{ccccc|c} 1 & 3 & 0 & -2 & 0 & 4 \\ 0 & 0 & 1 & -5 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 12 \end{array}\right)$$

is fully reduced. In general, if a linear system A**x**=b has the property that the matrix (A|b) is fully reduced, you can write down the solution set without any effort.

EXERCISE

Describe the set of solutions of this system

$$\begin{aligned} x_1 + \qquad\qquad - 2x_4 &= 4 \\ x_2 + 4x_3 - x_4 &= 0 \end{aligned}$$

$x_1 = 4 + 2x_4$
$x_2 = \quad x_4 - 4x_3$
x_3 = any real number
x_4 = any real number

The augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -2 & 4 \\ 0 & 1 & 4 & -1 & 0 \end{array}\right)$$

Whenever the reduced augmented augmented matrix has a column, other than the last, that does not contain a leading one, there are infinitely many solutions. In this case, columns 3 & 4 have this property.

Here are two more examples of systems for which (A|b) is reduced. In each case, the solution set can be described without any effort.

$$\begin{aligned} 1x_1 + 0x_2 + 0x_3 &= 17 \\ 0x_1 + 1x_2 + 0x_3 &= 1 \\ 0x_1 + 0x_2 + 1x_3 &= 2 \end{aligned}$$

This system has the *unique* solution: $(x_1,x_2,x_3) = (17,1,2)$.

$$\begin{aligned} 0x_1 + 1x_2 + 3x_3 &= 0 \\ 0x_1 + 0x_2 + 0x_3 &= 1 \end{aligned}$$

This system has *no solution* because the second equation cannot be satisfied.

These examples are important. ***Whenever a reduced system has a unique solution, it resembles the first example. When a reduced system has no solution, it has a row containing a one in the last column, like the second example.***

EXERCISE

What does a reduced system look like when it has a unique solution, and the number of equations is greater than the number of variables.

like this

$$\begin{aligned} 1x_1 + 0x_2 + 0x_3 &= 5 \\ 0x_1 + 1x_2 + 0x_3 &= 4 \\ 0x_1 + 0x_2 + 1x_3 &= 2 \\ 0x_1 + 0x_2 + 0x_3 &= 0 \end{aligned}$$

the bottom row is all 0s