School of Mathematics Georgia Institute of Technology

Matrix Inversion

An *identity matrix* has ones on the main diagonal, and zeros elsewhere. For example, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ **is the 2×2 identity matrix. If the product of two square matrices is the identity matrix, we say that those matrices are *inverses*. For example,**

$$I = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}\begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix}$$ **(check this!)**

so $\begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}$ **and** $\begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix}$ **are inverse matrices.**

EXERCISE

Does $\begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}$ also equal **I** ?

Although matrix multiplication is not commutative, it is a fact that whenever square matrices A and B are such that AB = I, then always BA = I also.

For any numbers a,b,c,d, notice that

$$I = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}.$$

Dividing by ad-bc, we get a useful formula:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

for the inverse of a 2×2 matrix. It is so useful that it is probably worth memorizing.

You can remember it this way: ***to invert a 2×2 matrix, switch the diagonal entries, negate the other entries, and divide by the determinant.***

EXERCISE

What is the inverse of

$$\begin{pmatrix} 2 & 3 \\ 6 & 8 \end{pmatrix} ?$$

answer:

$$\begin{pmatrix} -4 & 3/2 \\ 3 & -1 \end{pmatrix}$$

Not every square matrix has an inverse. Try using the formula to invert the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$ and see what happens! This is an example of a *non-invertible matrix*.

EXERCISE

Can you see why no matrix product $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$ can equal the identity matrix?

no matter how the a_{ij} are chosen, each row in the product will always be a multiple of (1 2).

The inverse of an n×n matrix A can be found (if it exists) by reducing the n×(2n) augmented matrix (A|I). If the reduced matrix has the form (I|B), then B is the inverse of A. If the reduced matrix does not have this form (that is, the first n columns are not the identity matrix) then A is not invertible.

EXERCISE

find the inverse of A =

$$\begin{pmatrix} 2 & -1 & 2 \\ -1 & 3 & -3 \\ 4 & 3 & 1 \end{pmatrix}$$

solution on next page

2	-1	2	|	1	0	0	This is the augmented matrix (A|I)
-1	3	-3	|	0	1	0	
4	3	1	|	0	0	1	

1	-.5	1	|	.5	0	0	
-1	3	-3	|	0	1	0	row#1 := (1/2)×row#1
4	3	1	|	0	0	1	

1	-.5	1	|	.5	0	0	
0	2.5	-2	|	.5	1	0	row#2 := row#2+row#1
4	3	1	|	0	0	1	

1	-.5	1	|	.5	0	0	
0	2.5	-2	|	.5	1	0	row#3 := row#3−4×row#1
0	5	-3	|	-2	0	1	

1	-.5	1	|	.5	0	0	
0	1	-.8	|	.2	.4	0	row#2 := (2/5)×row#2
0	5	-3	|	-2	0	1	

(continued)

1	0	.6	| .6	.2	0		
0	1	-.8	| .2	.4	0		row#1 := row#1 + (1/2)×row#2
0	5	-3	| -2	0	1		

1	0	.6	| .6	.2	0		
0	1	-.8	| .2	.4	0		row#3 := row#3 - 5×row#2
0	0	1	| -3	-2	1		

1	0	0	| 2.4	1.4	-.6		
0	1	-.8	| .2	.4	0		row#1 := row#1 - (3/5)×row#3
0	0	1	| -3	-2	1		

1	0	0	| 2.4	1.4	-.6		row#2 := row#2+(4/5)×row#3.
0	1	0	| -2.2	-1.2	.8		The righthand 3×3 matrix is
0	0	1	| -3	-2	1		the inverse of A.

2.4	1.4	-.6	
-2.2	-1.2	.8	is the inverse of A
-3	-2	1	