Gauss-Jordan Method

We have seen

1) how to reduce a matrix using the elementary row operations

2) how to describe the solution set of a linear system Ax=b whose augmented matrix (A|b) has been reduced.

Putting (1) and (2) together, we will have a systematic method for solving systems of linear equations.

Solving a linear system means determining the unique solution (when there is one), or deciding that no solution exists (when there isn't one), or finding a parametric description of the set of the complete solution set (when there are infinitely many).

The method works like this: **Given a system Ax=b, you reduce the augmented matrix (A|b). The reduced matrix (A′|b′) represents a new system of equations A′x = b′** ***having the same solution set (this is the key!).*** **Since (A′|b′) is reduced, we can simply write down its solution.**

Example: **for the system**

$$\begin{aligned} x_1 + 2x_2 &= 4 \\ 3x_1 - 4x_2 &= 2 \end{aligned}$$

we would form the augmented matrix $\left(\begin{array}{cc|c} 1 & 2 & 4 \\ 3 & -4 & 2 \end{array}\right)$ **which reduces to** $\left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right)$**. We can then write down the solution** $x_1=2$, $x_2=1$.

EXERCISE

Solve the system

$$\begin{aligned} 2x_1+5x_2&=19 \\ 4x_1-\ x_2&=\ 5 \end{aligned}$$

by the Gauss-Jordan method

The augmented matrix

$$\left(\begin{array}{cc|c} 2 & 5 & 19 \\ 4 & -1 & 5 \end{array}\right)$$

reduces to

$$\left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right)$$

Unique solution: $x_1=2$, $x_2=3$.

EXERCISE

Solve the system

$$2x_1 + 3x_2 = 8$$
$$4x_1 + 6x_2 = 16$$

by the Gauss-Jordan method

The augmented matrix

$$\left[\begin{array}{cc|c} 2 & 4 & 8 \\ 4 & 8 & 16 \end{array}\right]$$

reduces to

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 0 & 0 \end{array}\right]$$

There are infinitely many solutions:

$$x_1 = 4 - 2x_2$$
$$x_2 = \text{any real number}$$

EXERCISE

Solve the system

$$x_1 + 3x_2 = 4$$
$$3x_1 + 9x_2 = 10$$

by the Gauss-Jordan method

The augmented matrix

$$\left[\begin{array}{cc|c} 1 & 3 & 4 \\ 3 & 9 & 10 \end{array}\right]$$

reduces to

$$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

The bottom row says $0x_1+0x_2=1$, which is impossible. There is no solution.

Example: Let's solve the system

$$2x_1 - 3x_2 = -6$$
$$5x_1 + 4x_2 = 31$$

We form the matrix (A|b) and reduce it:

$$\left(\begin{array}{cc|c} 2 & -3 & -6 \\ 5 & 4 & 31 \end{array}\right)$$ This is the matrix (A|b)

$$\left(\begin{array}{cc|c} 1 & -3/2 & -3 \\ 5 & 4 & 31 \end{array}\right)$$ row#1 := 1/2×row#1

$$\left(\begin{array}{cc|c} 1 & -3/2 & -3 \\ 0 & 23/2 & 46 \end{array}\right)$$ row#2 := row#2 - 5×row#1

$$\left(\begin{array}{cc|c} 1 & -3/2 & -3 \\ 0 & 1 & 4 \end{array}\right)$$ row#2 := 2/23×row#

$$\left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 4 \end{array}\right)$$ row#1 := row#1 + 3/2×row#2

The reduced matrix represents the system

$$1x_1 + 0x_2 = 3$$
$$0x_1 + 1x_2 = 4$$

which has the unique solution $(x_1,x_2) = (3,4)$. This is also the unique solution to the original system.

Example: Let's solve the system

$$x_1 + x_2 + x_3 = 5$$
$$x_1 + 2x_2 + 3x_3 = 8$$

We form the matrix (A|b) and reduce it:

1	1	1	|	5	This is the matrix (A|b)
1	2	3	|	8	
1	1	1	|	5	row#2 := row#2 - row#1
0	1	2	|	3	the matrix is in echelon form
1	0	-1	|	2	row#1 := row#1- row#2
0	1	2	|	3	the matrix is reduced

The reduced matrix represents the system

$$1x_1 + 0x_2 - x_3 = 2$$
$$0x_1 + 1x_2 + 2x_3 = 3$$

which has infinitely many solutions:

$$x_1=2+x_3, \quad x_2=3-2x_3, \quad x_3=\text{any number}$$

We wrote the solution to the exercise on the previous page as two equations:

$$x_1 = 2 + x_3, \quad x_2 = 3 - 2x_3, \quad (x_3 \in R)$$

Notice that these two equations have the exact same meaning

as one vector equation: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$

When we write the equation this way, it becomes clear that the solution set is the line through the point (2,3,0) that is parallel to the vector (1,-2,1).

EXERCISE

In a previous exercise, we wrote a solution set as:

$$x_1 = 4 - 2x_2 \quad (x_2 \in R)$$

Rewrite this equation in vector form.

answer: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Homogeneous Systems

A system $Ax=b$ of linear equations is homogeneous if $b=0$. The solution set for a homogeneous system is never empty since $x=0$ is always a trivial solution. For a homogeneous system, there are only two possibilities: either there is a unique solution or there are infinitely many solutions.

For example, the system

$$3x_1+2x_2=0$$
$$x_1-5x_2=0$$

It is "homogeneous" because the numbers on the right-hand side are all zeros.

is homogeneous. Notice that $x=(0,0)$ is a "trivial" solution.

EXERCISE

Show that $x=(0,0)$ is the *only* solution to this homogeneous system. That is, there are no nontrival solutions

Solving in the usual way,

$$\left[\begin{array}{cc|c} 3 & 2 & 0 \\ 1 & -5 & 0 \end{array}\right] \text{ reduces to } \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

showing that $x_1=x_2=0$ is the only solution.

Example: **Taking** $A=\begin{pmatrix} 3 & -5 & -2 \\ 2 & 2 & -4 \end{pmatrix}$ **the system Ax=0 is**

$$3x_1-5x_2-2x_3=0$$
$$2x_1+2x_2-4x_3=0$$

Here is the row reduction:

3	-5	-2	|	0	Start with the matrix (A|b)
2	2	-4	|	0	
1	-5/3	-2/3	|	0	row#1 := 1/3 × row#1
2	2	-4	|	0	
1	-5/3	-2/3	|	0	row#2 := row#2 − 2 × row#1
0	16/3	-8/3	|	0	
1	-5/3	-2/3	|	0	row#2 := 3/16 × row#2
0	1	-1/2	|	0	
1	0	-2/3	|	0	row#1 := row#1 + 5/3 × row#2
0	1	-3/2	|	0	

notice the last column always contains zeros

EXERCISE

The row reduction has been done for you. Can you express the solution set for this homogeneous system in *two* different ways?

#1) $x_1=2x_3/3,\ x_2=3x_3/2,\ x_3\in\mathbb{R}$.

#2) $\mathbf{x}=x_3\begin{pmatrix} 2/3 \\ 3/2 \\ 1 \end{pmatrix}, x_3\in\mathbb{R}$, or $\mathbf{x}=t\begin{pmatrix} 4 \\ 9 \\ 6 \end{pmatrix}, t\in\mathbb{R}$

if you don't like fractions.

Two characteristics of homogeneous systems are worth pointing out in the example on the previous page.

First, during the row reduction process *the last column of the matrix always remains zero*. This always happens with homogeneous systems, and for this reason it is customary not to write down this column at all (but if you do that, remember to *keep it in mind that there is a phantom column of zeros* there).

Second, the solutions of a homogeneous system are sums of multiples of fixed vectors (there is no constant vector as there was in the nonhomogeneous case). In the example, the solutions were multiples of just one vector: $t\begin{pmatrix}4\\9\\6\end{pmatrix}$.