Basic matrix concepts

A ***matrix*** **is a collection of numbers arranged in rows and columns. For example, the matrix** $A = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 4 & 5 \end{pmatrix}$ **has two rows and three columns. It is therefore called a 2×3 matrix. (It would be an error to call it a 3×2 matrix – because the number of rows is always mentioned *before* the number of columns.) A matrix having only one row, such as** $\begin{pmatrix} 1 & 3 & 4 \end{pmatrix}$**, is a *row vector*. Likewise, a *column vector* such as** $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ **has only one column.**

But does it really make any difference whether a vector is written as a row or column?

Usually not at all, *but when it does matter, it matters a great deal*. When you multiply matrices it is very important to write the rows as rows and columns as columns.

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The numbers that occur in a matrix are called its *entries* or *components*. We refer to them by their row and column number, in that order. For example, the (2,1)-entry of the matrix A = $\begin{pmatrix} 2 & 1 & 0 \\ 3 & 4 & 5 \end{pmatrix}$ is 3. If the name of a matrix is a capital letter, like "A", it is customary to refer to its entries by the same letter, but in lower case. Thus the entries of A are $a_{11}=2$, $a_{12}=1$, $a_{13}=0$, $a_{21}=3$, $a_{22}=4$, $a_{23}=5$.

EXERCISE

$$\sum_{i=1}^{2} a_{i1} = ?$$

$$\sum_{i=1}^{2} a_{ii} = ?$$

$$\sum_{i=1}^{2} \sum_{j=1}^{3} a_{ij} = ?$$

answers: 5, 6, 15

Let $A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 5 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 1 & 4 & 2 \\ 0 & 6 & 2 & 8 \\ 1 & -1 & 3 & 0 \end{pmatrix}$, and $C = \begin{pmatrix} 2 & -1 \\ 3 & -2 \\ 0 & 1 \\ 2 & 5 \end{pmatrix}$.

Try to evaluate these expressions:

EXERCISE

$\sum_{j=1}^{3} a_{2j}b_{j4} = ?$ = 2*2+5*8+0*0=44

$\sum_{j=1}^{4} b_{2j}c_{j1} = ?$ = 0*2+6*3+2*0+8*2=34

$\sum_{j=1}^{2} c_{4j}a_{j3} = ?$ = 2*2+5*0=4

A vector with n components is an *n-dimensional vector*.

A vector can be a row vector or a column vector, so if I tell you that x is an n-dimensional vector without specifying whether x is a row or column vector, then which one should you believe it is: a row or a column?

In many cases it doesn't much matter, but sometimes it does, so the following convention is often adopted:

> ***Unless otherwise specified, a variable representing a vector always represents a column vector.***

This clears up any possible confusion. It is still possible to let a variable represent a row vector if you want. For example, you could write x=(7,3) if you want x to be a row, or $x=(7,3)^t$ if you want x to be a column.

The *transpose* of a matrix is the matrix you get by turning its rows into columns. For example, the transpose of $A = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 4 & 5 \end{pmatrix}$ is the matrix $A^t = \begin{pmatrix} 2 & 3 \\ 1 & 4 \\ 0 & 5 \end{pmatrix}$. Note the use of the superscript "t" to represent the transpose of a matrix.

EXERCISE

If you are told that the matrix A is equal to its transpose, what can you say about A ?

First of all, A would have to have the same number of rows and columns; that is A would be *square*. Second, the ij-entry and the ji-entry would have to be equal, for all choices of i and j.

If $A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 5 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 5 & 1 \\ 3 & 4 & 9 \end{pmatrix}$, then $A + B = \begin{pmatrix} 3 & 8 & 3 \\ 5 & 9 & 9 \end{pmatrix}$.

Also, $A^t = \begin{pmatrix} 1 & 2 \\ 3 & 5 \\ 2 & 0 \end{pmatrix}$, $B^t = \begin{pmatrix} 2 & 3 \\ 5 & 4 \\ 1 & 9 \end{pmatrix}$, and so $A^t + B^t = \begin{pmatrix} 3 & 5 \\ 8 & 9 \\ 3 & 9 \end{pmatrix}$.

This illustrates a basic rule:

for any matrices A and B (having the same number of rows and columns),

$$A^t + B^t = (A + B)^t$$

EXERCISE

try to simplify this expression:

$((A+B^t)^t - C^t)^t$

answer: $((A+B^t)^t - C^t)^t = A + B^t - C$

Special Matrices

There are a few matrices or types of matrices that we will encounter frequently, and which have special names.

A **zero matrix** is a matrix, all of whose entries are zeros. We use the symbol "0" to describe the zero matrix. What this actually means is that the symbol "0" can be used to represent *any* matrix of zeros. Thus it is equally correct

to write $0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ as it is to write $0 = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}$.

Although this may seem nonsensical, it is usually clear what is meant from the context in which the symbol "0" occurs.

EXERCISE

What does the symbol "0" mean in this context:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + 0 = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

It is the 2×3 zero matrix

A ***square matrix*** is a matrix having the same number of rows as columns. If B is an n×n square matrix, then the entries of the form b_{ii} are the ***diagonal entries*** of B. Thus $b_{11}=2$, $b_{22}=5$, and $b_{33}=3$ are the diagonal entries of $B=\begin{pmatrix} 2 & 1 & 4 \\ 7 & 5 & 6 \\ 9 & 8 & 3 \end{pmatrix}$. A square matrix, all of whose off-diagonal entries are zeros, is a ***diagonal matrix***. A diagonal matrix, all of whose diagonal entries are ones is called an ***identity matrix***.

EXERCISE

Which of the following matrices are diagonal?

a) $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ b) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 7 & 0 \\ 8 & 0 & 0 \end{pmatrix}$

c) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ d) $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{pmatrix}$

Answer:

(a) and (c) are diagonal.

A ***symmetric matrix*** is a square matrix A such that $a_{ij} = a_{ji}$ for all possible i and j. This is just a fancy way of saying that the matrix lands on top of itself when you flip all the entries across the diagonal. For example, $\begin{pmatrix} 5 & -6 & 2 \\ -6 & 1 & 3 \\ 2 & 3 & 4 \end{pmatrix}$ is a symmetric 3×3 matrix. A ***skew-symmetric*** matrix is a square matrix A such that $a_{ij} = -a_{ji}$ for all possible i and j. For example, $\begin{pmatrix} 0 & -4 & -7 \\ 4 & 0 & 3 \\ 7 & -3 & 0 \end{pmatrix}$ is a skew-symmetric 3×3 matrix.

EXERCISE

a) If the matrix A has the property that $A = A^t$, then what can you say about A ?

b) If the matrix A has the property that $A = -A^t$, then what can you say about A ?

a) A is symmetric

b) A is skew-symmetric