Three Proofs that TSM is Efficient

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Abstract

Many people do not understand why the cap-weighted total US stock market (TSM) plays such a central role in financial economics. They believe that TSM is just one of many possible US stock portfolios, with no good reason to believe that it is special or superior to other kinds of US stock portfolios. They often present alternatives which they claim offer a higher expected return than TSM with less risk. In technical terms, these alternatives are "more efficient" than TSM.

We give three proofs that, under three different assumptions, TSM is efficient, in the sense that no other US stock portfolio can be more efficient than TSM (have lower risk and higher expected return). The three assumptions are:

1. The Efficient Market Hypothesis.
2. The Capital Asset Pricing Model.
3. The Fama-French Three Factor Pricing Model.

If any one of these assumptions is true, TSM must be efficient. If any other US stock portfolio has a higher expected return than TSM with lower risk, we must reject all three of the assumptions.

1 Three Proofs that TSM is Efficient

Theorem 1 *If the Efficient Market Hypothesis is true, then TSM is efficient.*

Proof:

The Efficient Market Hypothesis (EMH) states that prices always reflect all available information, prices represent the wealth-weighted aggregate opinion of all investors as to the proper value of assets, and the current price of any asset is our best estimate of its fundamental value. Our assumption is that the US stock market is efficient.

The price of an asset is the amount of money it would take to buy it. In the context of the US stock market, an asset is a publicly owned US company, and its price is the current total market capitalization of the company's stock. Thus the current total market capitalization of a US company is our best estimate of its fundamental value.

Suppose for the purpose of a contradiction that some other US stock portfolio P is more efficient than TSM. P has a higher expected return than TSM and lower risk. P is clearly an unconditionally better portfolio than TSM. All investors should prefer P to TSM.

There must be some company C whose stock is weighted more heavily in P than in TSM. If this were not the case, P and TSM would be the same portfolio. Let p be the percentage of C's stock in portfolio P, let q be the percentage of C's stock in TSM, and let m be the current total market capitalization in dollars of the entire US stock market. Then we have:

$$p > q \tag{1}$$

Because of our assumption that the US stock market is efficient, the fundamental value of C is its current price, which is its current total market capitalization, which is $q * m$. So we have:

$$q * m = \text{fundamental value of } C \tag{2}$$

P is an unconditionally better portfolio than TSM, so all investors should prefer P to TSM. So all investors should prefer to hold p percent of C's stock in their portfolios rather than q percent. The sum of the total dollar values of all investor's portfolios is m. So investors in the wealth-weighted aggregate should prefer to hold a total of $p * m$ dollars of C's stock. The total number of dollars held in C's stock is C's total market capitalization, which is its price. So investors in the aggregate should prefer that the price of C is $p * m$. This equivalent to saying that the proper price or fundamental value of C is $p * m$. So we have:

$$p * m = \text{fundamental value of } C \tag{3}$$

(1), (2) and (3) are a contradiction. Therefore, P cannot be more efficient than TSM, and our theorem is proved.

Theorem 2 *If the Capital Asset Pricing Model is true, then TSM is efficient.*

Proof:

In the Capital Asset Pricing Model, "risk" is measured by a single number β and the risk and expected return of any asset or portfolio P are related by the following equation:

$$Rp - Rf = \beta(Rm - Rf) \tag{4}$$

where:

Rp = the expected return of portfolio P.
Rf = the risk-free rate of return.
β = the beta (risk) of portfolio P.
Rm = the expected return of TSM.

TSM is a portfolio, and if we let P = TSM, in equation (4) we get $Rp = Rm$, and we must have $\beta = 1$. So the beta of TSM is 1.

Suppose some other portfolio P has lower risk than TSM. Then the beta β of P is less than 1. By equation (4):

$$\begin{aligned} Rp - Rf &= \beta(Rm - Rf) \\ &< 1 \times (Rm - Rf) \\ &= Rm - Rf \\ Rp &< Rm \end{aligned}$$

We have shown that for any portfolio P, if P has lower risk than TSM, then P must also have a lower expected return than TSM. Therefore TSM is efficient.

Theorem 3 *If the Fama-French Three Factor Pricing Model is true, then TSM is efficient.*

Proof:

In the Fama-French model, risk is measured by three "beta" numbers for "market risk," "value risk" and "small risk."

A portfolio A is "less risky" than a portfolio B if all three of the risk betas for A are less than or equal to the corresponding risk betas of B, and at least one of A's risk betas is strictly less than the corresponding risk beta of B.

The expected return of a portfolio P is related to P's three risk betas by the following equation:

$$Rp - Rf = \beta 1 \times (Rm - Rf) + \beta 2 \times HML + \beta 3 \times SMB \tag{5}$$

where:

Rp = the expected return of portfolio P.
Rf = the risk-free rate of return.
$\beta 1$ = the market risk beta of P.
$\beta 2$ = the value risk beta of P.
$\beta 3$ = the small risk beta of P.
Rm = the expected return of TSM.
HML = a positive constant that measures the value stock premium.
SMB = a positive constant that measures the small stock premium.

In the Fama-French model, TSM has a market risk beta of 1 and value and size risk betas of 0.

Suppose some other portfolio P has lower risk than TSM. Then we must have:

$$\begin{aligned} \beta 1 &\leq 1 \\ \beta 2 &\leq 0 \\ \beta 3 &\leq 0 \end{aligned}$$

with strict inequality holding in at least one of the above inequalities.

By equation (5):

$$\begin{aligned} Rp - Rf &= \beta 1 \times (Rm - Rf) + \beta 2 \times HML + \beta 3 \times SMB \\ &< 1 \times (Rm - Rf) + 0 \times HML + 0 \times SMB \\ &= Rm - Rf \\ Rp &< Rm \end{aligned}$$

We have shown that for any portfolio P, if P has lower risk than TSM, then P must also have a lower expected return than TSM. Therefore TSM is efficient.