

The Normal and Lognormal Distributions

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Abstract

The basic properties of the normal and lognormal distributions, with full proofs.

We assume familiarity with elementary probability theory and with college-level calculus.

1 Definitions and Summary of the Propositions

Proposition 1: $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} dx = 1$

Proposition 2: $\int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} dx = \mu$

Proposition 3: $\int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} dx = \mu^2 + \sigma^2$

Definition 1 The normal distribution $N[\mu, \sigma^2]$ is the probability distribution defined by the following density function:

$$\frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$$

Note that Proposition 1 verifies that this is a valid density function (its integral from $-\infty$ to ∞ is 1).

Definition 2 The lognormal distribution $LN[\mu, \sigma^2]$ is the distribution of e^X where X is $N[\mu, \sigma^2]$.

Proposition 4: If X is $N[\mu, \sigma^2]$ then $E(X) = \mu$ and $\text{Var}(X) = \sigma^2$.

Proposition 5: If Y is $LN[\mu, \sigma^2]$ then $E(Y) = e^{\mu + \frac{1}{2}\sigma^2}$ and $\text{Var}(Y) = e^{2\mu + \sigma^2}(e^{\sigma^2} - 1)$.

Proposition 6: If X is $N[\mu, \sigma^2]$ then $aX + b$ is $N[a\mu + b, a^2\sigma^2]$.

Proposition 7: If X is $N[\mu_1, \sigma_1^2]$, Y is $N[\mu_2, \sigma_2^2]$, and X and Y are independent, then $X + Y$ is $N[\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2]$.

Corollary 1: If X_i are independent $N[\mu, \sigma^2]$ for $i = 1 \dots n$ then $\sum_{i=1}^n X_i$ is $N[n\mu, n\sigma^2]$.

Corollary 2: If Y_i are independent $LN[\mu, \sigma^2]$ for $i = 1 \dots n$ then $\prod_{i=1}^n Y_i$ is $LN[n\mu, n\sigma^2]$.

Proposition 8: The probability density function of $LN[\mu, \sigma^2]$ is:

$$\frac{1}{x\sqrt{2\pi}\sigma} e^{-(\log(x)-\mu)^2/2\sigma^2}$$

2 Proofs of the Propositions

Proposition 1

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} dx = 1$$

Proof:

First assume that $\mu = 0$ and $\sigma = 1$. Let:

$$a = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

Then:

$$\begin{aligned} a^2 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dx dy \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2} dx dy \end{aligned}$$

Apply the polar transformation $x = r \cos \theta$, $y = r \sin \theta$, $dx dy = r dr d\theta$:

$$\begin{aligned} a^2 &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} e^{-r^2/2} r dr d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left[-e^{-r^2/2} \right]_0^{\infty} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} [0 - (-1)] d\theta \\ &= \frac{1}{2\pi} 2\pi \\ &= 1 \end{aligned}$$

$a > 0$, and we just showed that $a^2 = 1$, so we must have $a = 1$.

For the general case, apply the transformation $y = \frac{x-\mu}{\sigma}$, $dy = \frac{dx}{\sigma}$:

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} dx &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/2} \sigma dy \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \\ &= 1 \end{aligned}$$

Proposition 2

$$\int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi\sigma}} e^{-(x-\mu)^2/2\sigma^2} dx = \mu$$

Proof:

First assume that $\mu = 0$ and $\sigma = 1$.

$$\begin{aligned} \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx &= \lim_{n \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \int_{-n}^n x e^{-x^2/2} dx \\ &= \frac{1}{\sqrt{2\pi}} \lim_{n \rightarrow \infty} \left[-e^{-x^2/2} \right]_{-n}^n \\ &= \frac{1}{\sqrt{2\pi}} \lim_{n \rightarrow \infty} [(-e^{-n^2/2}) - (-e^{-n^2/2})] \\ &= \frac{1}{\sqrt{2\pi}} \lim_{n \rightarrow \infty} 0 \\ &= 0 \end{aligned}$$

For the general case, apply the transformation $y = \frac{x-\mu}{\sigma}$, $dy = \frac{dx}{\sigma}$:

$$\begin{aligned} \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi\sigma}} e^{-(x-\mu)^2/2\sigma^2} dx &= \int_{-\infty}^{\infty} (\mu + \sigma y) \frac{1}{\sqrt{2\pi\sigma}} e^{-y^2/2} \sigma dy \\ &= \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy + \sigma \int_{-\infty}^{\infty} y \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \\ &= \mu \times 1 + \sigma \times 0 \quad (\text{by Proposition 1}) \\ &= \mu \end{aligned}$$

Proposition 3

$$\int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi\sigma}} e^{-(x-\mu)^2/2\sigma^2} dx = \mu^2 + \sigma^2$$

Proof:

First assume that $\mu = 0$ and $\sigma = 1$.

Integrate by parts using $f = x$, $f' = 1$, $g = -e^{-x^2/2}$, $g' = xe^{-x^2/2}$:

$$\begin{aligned} \int_{-n}^n x^2 e^{-x^2/2} dx &= \int_{-n}^n f g' dx \\ &= f(n)g(n) - f(-n)g(-n) - \int_{-n}^n f' g dx \\ &= (-ne^{-n^2/2}) - (ne^{-n^2/2}) - \int_{-n}^n -e^{-x^2/2} dx \end{aligned}$$

$$\begin{aligned}
&= -2ne^{-n^2/2} - \int_{-n}^n -e^{-x^2/2} dx \\
&= \int_{-n}^n e^{-x^2/2} dx - 2ne^{-n^2/2}
\end{aligned}$$

Then:

$$\begin{aligned}
\int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx &= \frac{1}{\sqrt{2\pi}} \lim_{n \rightarrow \infty} \int_{-n}^n x^2 e^{-x^2/2} dx \\
&= \frac{1}{\sqrt{2\pi}} \lim_{n \rightarrow \infty} \left[\int_{-n}^n e^{-x^2/2} dx - 2ne^{-n^2/2} \right] \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx - \lim_{n \rightarrow \infty} 2ne^{-n^2/2} \\
&= 1 - \lim_{n \rightarrow \infty} 2ne^{-n^2/2} \quad (\text{by Proposition 1})
\end{aligned}$$

All that remains is to show that the last limit above is 0. We do this using L'Hôpital's rule:

$$\lim_{n \rightarrow \infty} 2ne^{-n^2/2} = \lim_{n \rightarrow \infty} \frac{2n}{e^{n^2/2}} = \lim_{n \rightarrow \infty} \frac{2}{ne^{n^2/2}} = 0$$

For the general case, apply the transformation $y = \frac{x - \mu}{\sigma}$, $dy = \frac{dx}{\sigma}$:

$$\begin{aligned}
\int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} dx &= \int_{-\infty}^{\infty} (\mu + \sigma y)^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/2} \sigma dy \\
&= \mu^2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy + \\
&\quad 2\mu\sigma \int_{-\infty}^{\infty} y \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy + \\
&\quad \sigma^2 \int_{-\infty}^{\infty} y^2 \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \\
&= \mu^2 \times 1 + 2\mu\sigma \times 0 + \sigma^2 \times 1 \\
&\quad (\text{by Propositions 1 and 2}) \\
&= \mu^2 + \sigma^2
\end{aligned}$$

Proposition 4 *If X is $N[\mu, \sigma^2]$ then $E(X) = \mu$ and $\text{Var}(X) = \sigma^2$.*

Proof:

$$\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} dx = \mu \quad (\text{by Proposition 2}) \\
\text{Var}(X) &= E(X^2) - E(X)^2 \quad (\text{by Proposition 1 in reference [1]}) \\
&= \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} dx - \mu^2 \\
&= \mu^2 + \sigma^2 - \mu^2 \quad (\text{by Proposition 3}) \\
&= \sigma^2
\end{aligned}$$

Proposition 5 *If Y is $LN[\mu, \sigma^2]$ then $E(Y) = e^{\mu + \frac{1}{2}\sigma^2}$ and $\text{Var}(Y) = e^{2\mu + \sigma^2}(e^{\sigma^2} - 1)$.*

Proof:

$Y = e^X$ where X is $N[\mu, \sigma^2]$. First assume that $\mu = 0$:

$$\begin{aligned}
E(Y) &= E(e^X) = \int_{-\infty}^{\infty} e^x \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{2x\sigma^2 - x^2}{2\sigma^2}} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\sigma^2)^2 + \sigma^4}{2\sigma^2}} dx \\
&= e^{\frac{1}{2}\sigma^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\sigma^2)^2/2\sigma^2} dx \\
&= e^{\frac{1}{2}\sigma^2} \times 1 \quad (\text{by Proposition 1}) \\
&= e^{\frac{1}{2}\sigma^2} \\
E(Y^2) &= E(e^{2X}) = \int_{-\infty}^{\infty} e^{2x} \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2} dx \\
&= \int_{-\infty}^{\infty} e^{2x} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{4x\sigma^2 - x^2}{2\sigma^2}} dx \\
&= \int_{-\infty}^{\infty} e^{2x} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-2\sigma^2)^2 + 4\sigma^4}{2\sigma^2}} dx \\
&= e^{2\sigma^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-2\sigma^2)^2/2\sigma^2} dx \\
&= e^{2\sigma^2} \times 1 \quad (\text{by Proposition 1}) \\
&= e^{2\sigma^2} \\
\text{Var}(Y) &= E(Y^2) - E(Y)^2 \quad (\text{by Proposition 1 in reference [1]}) \\
&= e^{2\sigma^2} - (e^{\frac{1}{2}\sigma^2})^2 = e^{2\sigma^2} - e^{\sigma^2} = e^{\sigma^2}(e^{\sigma^2} - 1)
\end{aligned}$$

For the general case, apply the transformation $y = x - \mu$, $dy = dx$:

$$\begin{aligned}
E(Y) &= E(e^X) = \int_{-\infty}^{\infty} e^x \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} dx \\
&= \int_{-\infty}^{\infty} e^{\mu+y} \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/2\sigma^2} dy \\
&= e^{\mu} \int_{-\infty}^{\infty} e^y \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/2\sigma^2} dy \\
&= \text{(by the same reasoning as for the case } \mu = 0 \text{ above)} \\
&= e^{\mu} e^{\frac{1}{2}\sigma^2} \\
&= e^{\mu + \frac{1}{2}\sigma^2} \\
E(Y^2) &= E(e^{2X}) = \int_{-\infty}^{\infty} e^{2x} \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} dx \\
&= \int_{-\infty}^{\infty} e^{2(y+\mu)} \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/2\sigma^2} dy \\
&= e^{2\mu} \int_{-\infty}^{\infty} e^{2y} \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/2\sigma^2} dy \\
&= \text{(by the same reasoning as for the case } \mu = 0 \text{ above)} \\
&= e^{2\mu} e^{2\sigma^2} \\
&= e^{2\mu + 2\sigma^2} \\
\text{Var}(Y) &= E(Y^2) - E(Y)^2 \quad \text{(by Proposition 1 in reference [1])} \\
&= e^{2\mu + 2\sigma^2} - (e^{\mu + \frac{1}{2}\sigma^2})^2 \\
&= e^{2\mu + 2\sigma^2} - e^{2\mu + \sigma^2} \\
&= e^{2\mu + \sigma^2} (e^{\sigma^2} - 1)
\end{aligned}$$

Proposition 6 *If X is $N[\mu, \sigma^2]$ then $aX + b$ is $N[a\mu + b, a^2\sigma^2]$.*

Proof:

$$\begin{aligned}
\text{Prob}(aX + b < k) &= \text{Prob}(X < (k - b)/a) \\
&= \int_{-\infty}^{(k-b)/a} \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} dx \\
&= \text{(apply the transformation } y = ax + b, dy = adx) \\
&= \int_{-\infty}^k \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{y-b}{a} - \mu\right)^2/2\sigma^2} \frac{1}{a} dy \\
&= \int_{-\infty}^k \frac{1}{\sqrt{2\pi}(a\sigma)} e^{-(y-(a\mu+b))^2/2\sigma^2} dy
\end{aligned}$$

The last term above is the cumulative density function for $N[a\mu + b, a^2\sigma^2]$, so we have our result.

Proposition 7 *If X is $N[\mu_1, \sigma_1^2]$, Y is $N[\mu_2, \sigma_2^2]$, and X and Y are independent, then $X + Y$ is $N[\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2]$.*

Proof:

First assume that X is $N[0, 1]$ and Y is $N[0, \sigma^2]$. Then:

$$\begin{aligned}
 \text{Prob}(X + Y < k) &= \int \int_{x+u < k} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \frac{1}{\sqrt{2\pi\sigma}} e^{-u^2/2\sigma^2} dx du \\
 &= (\text{apply the transformation } u = \sigma y) \\
 &= \frac{1}{2\pi\sigma} \int \int_{x+\sigma y < k} e^{-(x^2+y^2)/2} \sigma dx dy \\
 &= \frac{1}{2\pi} \int \int_{x+\sigma y < k} e^{-(x^2+y^2)/2} dx dy \tag{1}
 \end{aligned}$$

At this point we temporarily make the assumption that $k \geq 0$. Figure 1 shows the area over which we are integrating. It is the half of the plane below and to the left of the line $x + \sigma y = k$.

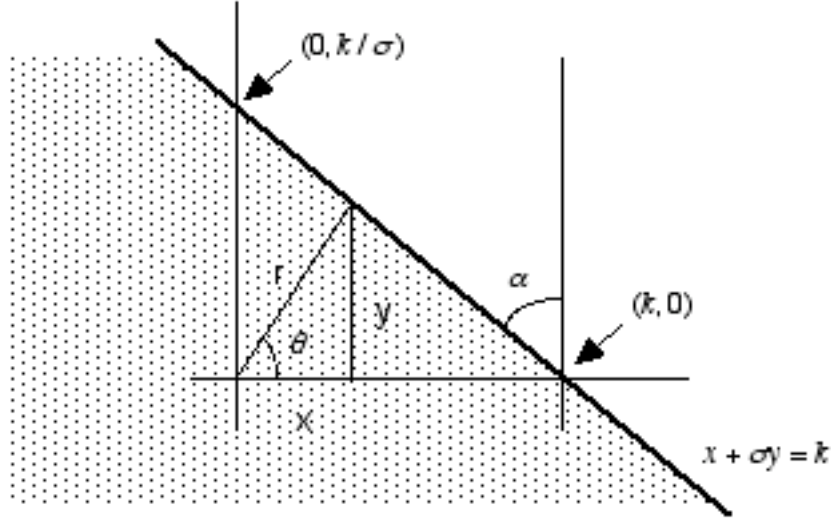


Figure 1: Area of Integration

Note the following relationships:

$$\begin{aligned}
 x &= r \cos \theta & r &= \frac{k}{\cos \theta + \sigma \sin \theta} \\
 y &= r \sin \theta & \tan \alpha &= k / (k/\sigma) = \sigma \\
 r \cos \theta + \sigma r \sin \theta &= x + \sigma y = k & \alpha &= \arctan \sigma
 \end{aligned}$$

We'll use the following technique to prove the result. First we'll convert the double integral above to polar coordinates. Then we'll rotate the result by $-\alpha$ so that the graph above becomes the one shown in 2 below. Then we'll show that the resulting integral is the same as the one for $\text{Prob}(Z < k)$ where Z is $N[0, 1 + \sigma^2]$.

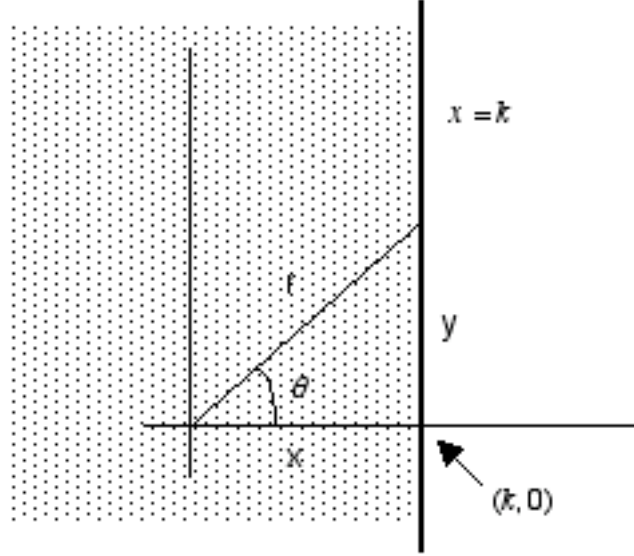


Figure 2: Area of Integration Rotated

Note that $r \cos \theta = x = k$, and $r = k / \cos \theta$.

Apply the polar transformation $x = r \cos \theta$, $y = r \sin \theta$, $dx dy = r dr d\theta$ to equation (1):

$$\begin{aligned}
 \text{Prob}(X + Y < k) &= \frac{1}{2\pi} \int \int_{x+\sigma y < k} e^{-(x^2+y^2)/2} dx dy \\
 &= \frac{1}{2\pi} \int \int_{x+\sigma y < k} r e^{-r^2/2} dr d\theta \\
 &= \frac{1}{2\pi} \int_{\pi/2+\alpha}^{3\pi/2+\alpha} \int_0^{\infty} r e^{-r^2/2} dr d\theta + \\
 &\quad \frac{1}{2\pi} \int_{-\pi/2+\alpha}^{3\pi/2+\alpha} \int_0^{\frac{k}{\cos \theta + \sigma \sin \theta}} r e^{-r^2/2} dr d\theta \quad (2)
 \end{aligned}$$

Note that we have made use of the assumption that $k \geq 0$ at this point to split the area over which we are integrating into two regions. In the first region, θ varies from $\pi/2 + \alpha$ to $3\pi/2 + \alpha$, and the vector at the origin with angle θ does not intersect the line $x + \sigma y$, so r varies from 0 to ∞ . In the second region, θ

varies from $-\pi/2 + \alpha$ to $\pi/2 + \alpha$, and the vector does intersect the line, so r varies from 0 to $k/(\cos \theta + \sigma \sin \theta)$.

We want to apply the transformation $\lambda = \theta - \alpha$ to rotate. We first must calculate what happens to the upper limit of integration in the last integral above under this transformation.

$$\begin{aligned} \frac{k}{\cos \theta + \sigma \sin \theta} &= \frac{k}{\cos(\lambda + \alpha) + \sigma \sin(\lambda + \alpha)} \\ &= \frac{k}{\cos(\lambda + \arctan \sigma) + \sigma \sin(\lambda + \arctan \sigma)} \end{aligned} \quad (3)$$

We now apply some trigonometric identities:

$$\begin{aligned} \cos(\lambda + \arctan \sigma) &= \cos \lambda \cos(\arctan \sigma) - \sin \lambda \sin(\arctan \sigma) \\ \sin(\lambda + \arctan \sigma) &= \sin \lambda \cos(\arctan \sigma) + \cos \lambda \sin(\arctan \sigma) \\ \cos(\arctan \sigma) &= \frac{1}{\sqrt{1 + \sigma^2}} \\ \sin(\arctan \sigma) &= \frac{\sigma}{\sqrt{1 + \sigma^2}} \\ \cos(\lambda + \arctan \sigma) &= \frac{\cos \lambda - \sigma \sin \lambda}{\sqrt{1 + \sigma^2}} \end{aligned} \quad (4)$$

$$\sigma \sin(\lambda + \arctan \sigma) = \frac{\sigma \sin \lambda + \sigma^2 \cos \lambda}{\sqrt{1 + \sigma^2}} \quad (5)$$

Adding equations (4) and (5) gives:

$$\begin{aligned} \cos(\lambda + \arctan \sigma) + \sigma \sin(\lambda + \arctan \sigma) &= \frac{\cos \lambda + \sigma^2 \cos \lambda}{\sqrt{1 + \sigma^2}} \\ &= \frac{\cos \lambda(1 + \sigma^2)}{\sqrt{1 + \sigma^2}} \\ &= \cos \lambda \sqrt{1 + \sigma^2} \end{aligned} \quad (6)$$

We can now do our rotation under the transformation $\lambda = \theta - \alpha$. Equations (2), (3) and (6) give:

$$\begin{aligned} \text{Prob}(X + Y < k) &= \frac{1}{2\pi} \int_{\pi/2}^{3\pi/2} \int_0^\infty r e^{-r^2/2} dr d\lambda + \\ &\quad \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \int_0^{\frac{k}{\cos \lambda \sqrt{1 + \sigma^2}}} r e^{-r^2/2} dr d\lambda \\ &= \frac{1}{2\pi} \int_{\pi/2}^{3\pi/2} \left[-e^{-r^2/2} \right]_0^\infty d\lambda + \\ &\quad \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \left[-e^{-r^2/2} \right]_0^{\frac{k}{\cos \lambda \sqrt{1 + \sigma^2}}} d\lambda \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2\pi} \int_{\pi/2}^{3\pi/2} [1] d\lambda + \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \left[-e^{\frac{-k^2}{2\cos^2\lambda(1+\sigma^2)}} - (-1) \right] d\lambda \\
&= 1 - \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{\frac{-k^2}{2\cos^2\lambda(1+\sigma^2)}} d\lambda \tag{7}
\end{aligned}$$

Now we turn our attention to evaluating $\text{Prob}(Z < k)$ where Z is $N[0, 1 + \sigma^2]$. Let W be another random variable which is also $N[0, 1 + \sigma^2]$. We use a sequence of steps similar to the one above, only without the rotation:

$$\begin{aligned}
\text{Prob}(Z < k) &= \text{Prob}(Z < k, -\infty < W < \infty) \\
&= \int \int_{z < k} \frac{1}{\sqrt{2\pi}\sqrt{1+\sigma^2}} e^{-z^2/2(1+\sigma^2)} \frac{1}{\sqrt{2\pi}\sqrt{1+\sigma^2}} e^{-w^2/2(1+\sigma^2)} dz dw \\
&= \frac{1}{2\pi(1+\sigma^2)} \int \int_{z < k} e^{-(z^2+w^2)/2(1+\sigma^2)} dz dw \\
&= \frac{1}{2\pi(1+\sigma^2)} \int \int_{z < k} r e^{-r^2/2(1+\sigma^2)} dr d\theta \\
&= \frac{1}{2\pi(1+\sigma^2)} \int_{\pi/2}^{3\pi/2} \int_0^\infty r e^{-r^2/2(1+\sigma^2)} dr d\theta + \\
&\quad \frac{1}{2\pi(1+\sigma^2)} \int_{-\pi/2}^{\pi/2} \int_0^{k/\cos\theta} r e^{-r^2/2(1+\sigma^2)} dr d\theta \\
&= \frac{1}{2\pi(1+\sigma^2)} \int_{\pi/2}^{3\pi/2} \left[-(1+\sigma^2) e^{-r^2/2(1+\sigma^2)} \right]_0^\infty d\theta + \\
&\quad \frac{1}{2\pi(1+\sigma^2)} \int_{-\pi/2}^{\pi/2} \left[-(1+\sigma^2) e^{-r^2/2(1+\sigma^2)} \right]_0^{k/\cos\theta} d\theta \\
&= 1 - \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{\frac{-k^2}{2\cos^2\theta(1+\sigma^2)}} d\theta \tag{8}
\end{aligned}$$

Equations (7) and (8) are the same, so at this point we have completed our proof that $\text{Prob}(X + Y < k) = \text{Prob}(Z < k)$ when $k \geq 0$.

Now suppose that $k < 0$. Proposition 6 implies that for any normal random variable X with mean 0, $-X$ is also normally distributed with mean 0 and the same variance as X . Let $A = -X$, $B = -Y$, and $W = -Z$. Then A is $N[0, 1]$, B is $N[0, \sigma^2]$, and W is $N[0, 1 + \sigma^2]$. So we have:

$$\begin{aligned}
\text{Prob}(X + Y < k) &= \text{Prob}(-(X + Y) > -k) \\
&= \text{Prob}(A + B > -k) \\
&= 1 - \text{Prob}(A + B < -k) \\
&= 1 - \text{Prob}(W < -k) \quad (\text{because } -k > 0) \\
&= \text{Prob}(W > -k) \\
&= \text{Prob}(-Z > -k) \\
&= \text{Prob}(Z < k)
\end{aligned}$$

At this point we have shown that $\text{Prob}(X + Y < k) = \text{Prob}(Z < k)$ for all k . Thus the random variables $X + Y$ and Z have the same cumulative density function. Z is $N[0, 1 + \sigma^2]$, so $X + Y$ is also $N[0, 1 + \sigma^2]$.

This completes our proof for the case that X is $N[0, 1]$ and Y is $N[0, \sigma^2]$.

For the general case where X is $N[\mu_1, \sigma_1^2]$ and Y is $N[\mu_2, \sigma_2^2]$, let $A = (X - \mu_1)/\sigma_1$ and $B = (Y - \mu_2)/\sigma_1$. By Property 6, A is $N[0, 1]$ and B is $N[0, \sigma_2^2/\sigma_1^2]$. Thus:

$$\begin{aligned} \text{Prob}(X + Y < k) &= \text{Prob}(A + B < (k - \mu_1 - \mu_2)/\sigma_1) \\ &= \int_{-\infty}^{(k - \mu_1 - \mu_2)/\sigma_1} \frac{1}{\sqrt{2\pi}\sqrt{1 + \sigma_2^2/\sigma_1^2}} e^{-x^2/2(1 + \sigma_2^2/\sigma_1^2)} dx \\ &= (\text{apply the transformation } y = \sigma_1 x + \mu_1 + \mu_2) \\ &\quad \int_{-\infty}^k \frac{1}{\sqrt{2\pi}\sqrt{\sigma_1^2 + \sigma_2^2}} e^{-(y - (\mu_1 + \mu_2))^2/2(\sigma_1^2 + \sigma_2^2)} dy \end{aligned}$$

This is the cumulative density function for $N[\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2]$, so we have our full result.

Corollary 1 *If X_i are independent $N[\mu, \sigma^2]$ for $i = 1 \dots n$ then $\sum_{i=1}^n X_i$ is $N[n\mu, n\sigma^2]$.*

Proof:

This corollary follows immediately from Proposition 7.

Corollary 2 *If Y_i are independent $LN[\mu, \sigma^2]$ for $i = 1 \dots n$ then $\prod_{i=1}^n Y_i$ is $LN[n\mu, n\sigma^2]$.*

Proof:

This corollary follows immediately from Definition 2 and Proposition 7.

Proposition 8 *The probability density function of $LN[\mu, \sigma^2]$ is:*

$$\frac{1}{x\sqrt{2\pi}\sigma} e^{-(\log(x)-\mu)^2/2\sigma^2}$$

Proof:

Suppose X is $LN[\mu, \sigma^2]$. Then $X = e^Y$ where Y is $N[\mu, \sigma^2]$. Then:

$$\begin{aligned} \text{Prob}(X < k) &= \text{Prob}(e^Y < k) \\ &= \text{Prob}(Y < \log(k)) \\ &= \int_{-\infty}^{\log(k)} \frac{1}{\sqrt{2\pi}\sigma} e^{-(y-\mu)^2/2\sigma^2} dy \\ &= \text{(apply the transformation } x = e^y, y = \log(x), dy = \frac{1}{x} dx) \\ &\quad \int_{-\infty}^k \frac{1}{x\sqrt{2\pi}\sigma} e^{-(\log(x)-\mu)^2/2\sigma^2} dx \end{aligned}$$

References

- [1] John Norstad. Probability review.
<http://homepage.mac.com/j.norstad/finance>, Sep 2002.