

# An Exact Equation for the Median Return

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## **Abstract**

A short derivation of the exact equation for computing the median or “annualized” return of a lognormal random walk asset, given the simply compounded average or “expected” return and standard deviation of the simply compounded returns.

## 1 An Exact Equation for the Median Return

Let  $Y$  be a lognormal random variable giving the ending value after one year of a \$1 investment in an asset. Let:

- $\mu$  = the expected continuously compounded return of the asset.
- $\sigma$  = the standard deviation of the continuously compounded returns.

By Proposition 5 in reference [1] we have:

$$\begin{aligned} \mathbf{E}(Y) &= e^{\mu + \frac{1}{2}\sigma^2} \\ \mathbf{Var}(Y) &= e^{2\mu + \sigma^2}(e^{\sigma^2} - 1) \end{aligned}$$

Let:

- $r$  = the simply compounded expected return of the asset.
- $s$  = the standard deviation of the simply compounded returns.

Then:

$$\begin{aligned} 1 + r &= \mathbf{E}(Y) \\ &= e^{\mu + \frac{1}{2}\sigma^2} \\ s^2 &= \mathbf{Var}(Y) \\ &= e^{2\mu + \sigma^2}(e^{\sigma^2} - 1) \\ &= \left(e^{\mu + \frac{1}{2}\sigma^2}\right)^2 (e^{\sigma^2} - 1) \\ &= (1 + r)^2 (e^{\sigma^2} - 1) \end{aligned}$$

We can solve these equations for  $\sigma^2$  and  $\mu$ :

$$\begin{aligned} \sigma^2 &= \log\left(\frac{s^2}{(1+r)^2} + 1\right) \\ \mu &= \log(1+r) - \frac{1}{2}\sigma^2 \\ &= \log(1+r) - \frac{1}{2}\log\left(\frac{s^2}{(1+r)^2} + 1\right) \\ &= \log(1+r) - \log\left(\sqrt{\frac{s^2}{(1+r)^2} + 1}\right) \\ &= \log\left(\frac{1+r}{\sqrt{\frac{s^2}{(1+r)^2} + 1}}\right) \\ &= \log\left(\frac{(1+r)^2}{\sqrt{s^2 + (1+r)^2}}\right) \end{aligned}$$

Let:

$a$  = the median yearly simply compounded return

Then:

$$\begin{aligned} a &= e^\mu - 1 \\ &= e^{\log\left(\frac{(1+r)^2}{\sqrt{s^2+(1+r)^2}}\right)} - 1 \\ &= \frac{(1+r)^2}{\sqrt{s^2+(1+r)^2}} - 1 \end{aligned}$$

This is our exact equation for the median return.

If we take the Taylor series expansion of this equation about  $s = 0$ , the first three terms give the following approximation:

$$a \approx r - \frac{\frac{1}{2}s^2}{1+r}$$

This is similar to, but a bit more accurate than, the frequently seen approximation:

$$a \approx r - \frac{1}{2}s^2$$

## **References**

- [1] John Norstad. The normal and lognormal distributions.  
<http://homepage.mac.com/j.norstad/finance>, Feb 1999.