Linear Algebra Review

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Abstract

A review of the notions of matrix singularity and solutions to simultaneous systems of linear equations, with proofs.

1 Definitions and Summary of the Propositions

Proposition 1: Let V be an $n \times n$ matrix of values $v_{i,j}$. $Vw = 0$ for some column vector $w \neq 0$ iff $Vw = c$ has no solution for some column vector c.

Definition 1 *A square matrix* V *is* singular *iff* $Vw = 0$ *for some column vector* $w \neq 0$ *or, equivalently (by Proposition 1), iff* $Vw = c$ *has no solution for some column vector* c*.*

Definition 2 *The* $n \times n$ identity matrix I *is the matrix with with values:*

$$i_{a,b} = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}$$

Definition 3 *For an* $n \times n$ *matrix* V *an* inverse *of* V *is an* $n \times n$ *matrix* V^{-1} *which satisfies the following property:*

$$V \times V^{-1} = V^{-1} \times V = I$$

where I *is the* $n \times n$ *identity matrix.*

Proposition 2: A square matrix V has an inverse V^{-1} iff it is non-singular. In this case, the solution to the simultaneous set of linear equations $Vw = c$ is $w = V^{-1}c$.

2 Proofs of the Propositions

We begin with a notational convention we will use throughout this section.

Let V be an $n \times n$ matrix of values $v_{i,j}$ with a non-zero element $v_{a,b}$. Define the $(n-1) \times (n-1)$ matrix $\tilde{V}$ by eliminating row a and column b as follows:

$$\tilde{v}_{i,j} = v_{i,j} - \frac{v_{i,b} v_{a,j}}{v_{a,b}} \qquad \text{for all } i \neq a \text{ and } j \neq b$$

This notation is a bit goofy. $\tilde{V}$ is an $(n-1) \times (n-1)$ matrix, but we are still numbering the rows and columns using indices $1 \ldots n$, with row $i = a$ and column $j = b$ missing. Thus the rows are numbered $i = 1, \ldots, a-1, a+1, \ldots, n$ and the columns are numbered $j = 1, \ldots, b-1, b+1, \ldots, n$.

Lemma 1 *$Vw = 0$ for some column vector $w \neq 0$ iff $\tilde{V}\tilde{w} = 0$ for some column vector $\tilde{w} \neq 0$.*

Proof:

Suppose $Vw = 0$ for some column vector $w \neq 0$:

$$\begin{pmatrix} v_{1,1} & \cdots & v_{1,n} \\ \vdots & & \vdots \\ v_{n,1} & \cdots & v_{n,n} \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

We first show that we must have $w_i \neq 0$ for some $i \neq b$. Suppose this is not true, and $w_i = 0$ for all $i \neq b$. Row a of the equation becomes:

$$0 = \sum_{j=1}^{n} v_{a,j} w_j = v_{a,b} w_b$$

We know that $v_{a,b} \neq 0$, so we must have $w_b = 0$. But then $w_i = 0$ for all i, contrary to our assumption that $w \neq 0$.

Define $\tilde{w}_i = w_i$ for all $i \neq b$. We have just shown that the vector $\tilde{w} \neq 0$. Row i of $\tilde{V}\tilde{w}$ is:

$$\begin{aligned} \sum_{\substack{j=1 \\ j \neq b}}^{n} \tilde{v}_{i,j} \tilde{w}_j &= \sum_{\substack{j=1 \\ j \neq b}}^{n} \left(v_{i,j} - \frac{v_{i,b} v_{a,j}}{v_{a,b}} \right) w_j \\ &= \sum_{j=1}^{n} \left(v_{i,j} - \frac{v_{i,b} v_{a,j}}{v_{a,b}} \right) w_j - \left(v_{i,b} - \frac{v_{i,b} v_{a,b}}{v_{a,b}} \right) \\ &= \sum_{j=1}^{n} v_{i,j} w_j - \frac{v_{i,b}}{v_{a,b}} \sum_{j=1}^{n} v_{a,b} w_j - \left(v_{i,b} - \frac{v_{i,b} v_{a,b}}{v_{a,b}} \right) \\ &= 0 - 0 - 0 \\ &= 0 \end{aligned}$$

We have shown that $\tilde{V}\tilde{w} = 0$ for some $\tilde{w} \neq 0$. This completes one direction of the proof.

For the other direction, suppose $\tilde{V}\tilde{w} = 0$ for some $\tilde{w} \neq 0$. Define:

$$w_i = \begin{cases} \tilde{w}_i & \text{if } i \neq b \\ -\displaystyle\sum_{\substack{j=1 \\ j\neq b}}^{n} \frac{v_{a,j}}{v_{a,b}} \tilde{w}_j & \text{if } i = b \end{cases}$$

Row i of Vw is:

$$\begin{aligned}
\sum_{j=1}^{n} v_{i,j} w_j &= v_{i,b} w_b + \sum_{\substack{j=1 \\ j\neq b}}^{n} v_{i,j} w_j \\
&= v_{i,b} \left(-\sum_{\substack{j=1 \\ j\neq b}}^{n} \frac{v_{a,j}}{v_{a,b}} \tilde{w}_j \right) + \sum_{\substack{j=1 \\ j\neq b}}^{n} v_{i,j} \tilde{w}_j \\
&= \sum_{\substack{j=1 \\ j\neq b}}^{n} \left(v_{i,j} - \frac{v_{i,b} v_{a,j}}{v_{a,b}} \right) \tilde{w}_j \\
&= \text{for } i \neq a \quad \sum_{\substack{j=1 \\ j\neq b}}^{n} \tilde{v}_{i,j} \tilde{w}_j = 0 \\
&= \text{for } i = a \quad \sum_{\substack{j=1 \\ j\neq b}}^{n} \left(v_{a,j} - \frac{v_{a,b} v_{a,j}}{v_{a,b}} \right) \tilde{w}_j = 0
\end{aligned}$$

So $Vw = 0$ and our proof is complete.

Lemma 2 *$Vw = c$ has a solution for all column vectors c iff $\tilde{V}\tilde{w} = \tilde{c}$ has a solution for all column vectors $\tilde{c}$.*

Proof:

First suppose $Vw = c$ has a solution for all column vectors c and we are given a vector $\tilde{c}$. Define the column vector c as:

$$c_i = \begin{cases} \tilde{c}_i & \text{if } i \neq a \\ 0 & \text{if } i = a \end{cases}$$

Let w be a solution to $Vw = c$. Define:

$$\tilde{w}_i = w_i \quad \text{for } i \neq b$$

Row i of $\tilde{V}\tilde{w}$ is:

$$\begin{aligned}
\sum_{\substack{j=1\\ j\neq b}}^{n} \tilde{v}_{i,j}\tilde{w}_j &= \sum_{\substack{j=1\\ j\neq b}}^{n} \left(v_{i,j} - \frac{v_{i,b}v_{a,j}}{v_{a,b}}\right)\tilde{w}_j \\
&= \sum_{j=1}^{n} \left(v_{i,j} - \frac{v_{i,b}v_{a,j}}{v_{a,b}}\right)w_j - \left(v_{i,b} - \frac{v_{i,b}v_{a,b}}{v_{a,b}}\right)w_b \\
&= \sum_{j=1}^{n} \left(v_{i,j} - \frac{v_{i,b}v_{a,j}}{v_{a,b}}\right)w_j - 0 \\
&= \sum_{j=1}^{n} v_{i,j}w_j - \frac{v_{i,b}}{v_{a,b}}\sum_{j=1}^{n} v_{a,j}w_j \\
&= c_i - \frac{v_{i,b}}{v_{a,b}}c_a \\
&= c_i \\
&= \tilde{c}_i
\end{aligned}$$

So $\tilde{V}\tilde{w} = \tilde{c}$ has a solution for all column vectors $\tilde{c}$ and the first half of our proof is complete.

For the other direction suppose $\tilde{V}\tilde{w} = \tilde{c}$ has a solution for all column vectors $\tilde{c}$ and we are given a column vector c. Define:

$$\tilde{c}_i = c_i - \frac{c_a v_{i,b}}{v_{a,b}} \qquad \text{for } i \neq a$$

Let $\tilde{w}$ be a solution to $\tilde{V}\tilde{w} = \tilde{c}$. Define:

$$w_i = \begin{cases} \tilde{w}_i & \text{if } i \neq b \\ \dfrac{c_a}{v_{a,b}} - \displaystyle\sum_{\substack{j=1\\ j\neq b}}^{n} \frac{v_{a,j}}{v_{a,b}}\tilde{w}_j & \text{if } i = b \end{cases}$$

Row i of Vw is:

$$\begin{aligned}
\sum_{j=1}^{n} v_{i,j}w_j &= v_{i,b}w_b + \sum_{\substack{j=1\\ j\neq b}}^{n} v_{i,j}w_j \\
&= v_{i,b}\left(\frac{c_a}{v_{a,b}} - \sum_{\substack{j=1\\ j\neq b}}^{n} \frac{v_{a,j}}{v_{a,b}}\tilde{w}_j\right) + \sum_{\substack{j=1\\ j\neq b}}^{n} v_{i,j}w_j \\
&= \frac{c_a v_{i,b}}{v_{a,b}} + \sum_{\substack{j=1\\ j\neq b}}^{n} \left(v_{i,j} - \frac{v_{i,b}v_{a,j}}{v_{a,b}}\right)w_j
\end{aligned}$$

$$\begin{aligned} &= \quad \text{for } i \neq a \quad \frac{c_a v_{i,b}}{v_{a,b}} + \sum_{\substack{j=1\\ j\neq b}}^{n} \tilde{v}_{i,j} w_j = \frac{c_a v_{i,b}}{v_{a,b}} + \tilde{c}_i = c_i \\ &= \quad \text{for } i = a \quad \frac{c_a v_{a,b}}{v_{a,b}} + \sum_{\substack{j=1\\ j\neq b}}^{n} \left(v_{a,j} - \frac{v_{a,b} v_{a,j}}{v_{a,b}} \right) w_j = c_a \end{aligned}$$

So $Vw = c$ has a solution for all column vectors c and our proof is complete.

Proposition 1 *Let* V *be an* $n \times n$ *matrix of values* $v_{i,j}$. $Vw = 0$ *for some column vector* $w \neq 0$ *iff* $Vw = c$ *has no solution for some column vector* c.

Proof:

Our proof is by induction on the matrix dimension n.

The case $n = 1$ is trivial. V is the matrix $(v_{1,1})$ and the two conditions hold iff $v_{1,1} = 0$.

Suppose the theorem is true for all square matrices with dimension $n - 1$ and let V be a square matrix of dimension n.

If all elements of V are 0 the theorem is trivial because both conditions are clearly true. So we may assume that V has an element $v_{a,b} \neq 0$. Define the $(n-1) \times (n-1)$ matrix $\tilde{V}$ as above. Then:

$Vw = 0$ for some $w \neq 0$
iff $\tilde{V}\tilde{w} = 0$ for some $\tilde{w} \neq 0$ (by Lemma 1)
iff $\tilde{V}\tilde{w} = \tilde{c}$ has no solution for some $\tilde{c}$ (by the induction hypotheses)
iff $Vw = c$ has no solution for some c (by Lemma 2)

Proposition 1 says the V is non-singular if and only if all simultaneous linear equations with coefficients defined by V have solutions, which in turn is true if and only if the column vectors of V are linearly independent.

Proposition 2 *A square matrix V has an inverse V^{-1} iff it is non-singular. In this case, the solution to the simultaneous set of linear equations $Vw = c$ is $w = V^{-1}c$.*

Proof:

Suppose V has an inverse V^{-1} and let c be any column vector. Define $w = V^{-1}c$. Then $Vw = V(V^{-1}c) = (VV^{-1})c = Ic = c$. Thus $Vw = c$ has a solution for all c, and V is non-singular by Definition **??**.

Now suppose V is non-singular. For each i let c^i be the column vector with values:

$$c^i_a = \begin{cases} 1 & \text{if } a = i \\ 0 & \text{if } a \neq i \end{cases}$$

Because V is non-singular, $Vw^i = c^i$ has a solution for all i. Let V^{-1} be the matrix defined by setting column i of V^{-1} to w^i. Then:

$$V \times V^{-1} = V \times \big((w^1) \cdots (w^n)\big) = \big((c^1) \cdots (c^n)\big) = I$$

We complete the proof by showing that $V^{-1} \times V = I$:

$$\begin{aligned} V \times V^{-1} &= I \\ V^{-1} \times \big(V \times V^{-1}\big) \times V &= V^{-1} \times (I) \times V \\ \big(V^{-1} \times V\big) \times \big(V^{-1} \times V\big) &= \big(V^{-1} \times I\big) \times V \\ (I) \times (I) &= V^{-1} \times V \\ I &= V^{-1} \times V \end{aligned}$$