

# Black-Scholes the Easy Way

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## Abstract

In which we jump into our tardis and travel to a strange alternate universe where investors are risk-neutral instead of risk-averse. Surprisingly, in this universe the Black-Scholes equation for European option pricing is exactly the same as in our own universe. It's also ridiculously easy to derive. All we need is regular calculus – one simple evaluation of a definite integral derives the equation. We don't need any of the fancy machinery of stochastic calculus, arbitrage argument tricks, Ito's lemma, the heat equation, or parabolic differential equations that we use to derive Black-Scholes in our own universe.

## 1 Introduction

In [2] we briefly discussed why the price (value) of a volatile asset is less than its expected value. We also saw how the option pricing formula for trivial two-state options is the same in an imaginary risk-neutral universe as it is in our own risk-averse universe. In this note we explore the same phenomenon for real-life options and the Black-Scholes equation.

The reason why asset prices are less than their expected values in our universe is that when everything else is equal, investors prefer safe (risk-free) investments to volatile (risky) investments. This is called “risk-aversion.” Before an investor will buy a risky investment, he must be convinced that the expected return on that investment will exceed what he could earn by simply putting his money in the bank. The difference between the risk-free interest rate offered by the bank and the expected return on the risky investment is called the “risk premium.” The greater the volatility of an asset, the greater its risk premium.

One can imagine a universe in which investors don’t care about risk, and all they care about is expected return. Two investments with the same expected return are equally attractive in this universe, even if one is perfectly safe (e.g., a bank), and one is very risky (e.g., stocks). Investors in this imaginary universe are called “risk-neutral.” This universe has no risk premia. In this universe, the value of a volatile asset is its expected future value discounted to present value using the risk-free rate.

Clearly assets are priced very differently in our normal risk-averse universe than they are in this imaginary risk-neutral universe.

With options and other financial derivatives, however, it turns out that the equations which relate the values of the derivatives to the values of their underlying assets are exactly the same in the risk-neutral universe as they are in our risk-averse universe! This important property is called “the principle of risk-neutral valuation.” The principle tells us that when we are dealing with a derivative, we can compute its value as a function of the value of the underlying asset by making the assumption that we are in a risk-neutral universe.

This assumption can be powerful. In some cases (but not all), computing the value of a derivative as its discounted expected value is quite easy, while computing the same equation in our risk-averse universe can be very complicated indeed.

As an example, in this note we give a full derivation of the Black-Scholes equation for European call option pricing using the principle of risk-neutral valuation. The derivation involves a single relatively simple evaluation of a definite integral. The only tools required are elementary probability theory and integral calculus.

## 2 Derivation

**Definition 1** A universe is “risk-neutral” if for all assets  $A$  and all time periods  $t$ , the value of the asset  $V(A, 0)$  at time  $t = 0$  is the expected value of the asset at time  $t$  discounted to its present value using the risk-free rate. The equation is:

$$V(A, 0) = e^{-rt} E(V(A, t))$$

where  $r$  is the continuously compounded risk-free rate and  $V(A, t)$  is a random variable giving the asset value at time  $t$ .

**Lemma 1** In a risk-neutral universe, if the value of an asset follows the random walk  $\frac{ds}{s} = e^{\mu dt + \sigma dX} - 1$  where  $dX$  is  $N[0, dt]$ , and if  $r$  is the continuously compounded risk-free rate, then:

$$r = \mu + \frac{1}{2}\sigma^2$$

**Proof:**

By Theorem 4.1 in [3]:

$$s(t) = s(0)e^{\mu t + \sigma X} \quad \text{where } X \text{ is } N[0, t]$$

$s(t)$  is  $LN[\log(s(0)) + \mu t, \sigma^2 t]$ , so by Proposition 5 in [1]:

$$E(s(t)) = s(0)e^{(\mu + \frac{1}{2}\sigma^2)t}$$

Because the universe is risk-neutral, by Definition 1 above we have:

$$s(0) = e^{-rt} E(s(t)) = e^{-rt} s(0)e^{(\mu + \frac{1}{2}\sigma^2)t}$$

Thus:

$$\begin{aligned} 1 &= e^{-rt} e^{(\mu + \frac{1}{2}\sigma^2)t} \\ e^{rt} &= e^{(\mu + \frac{1}{2}\sigma^2)t} \\ rt &= (\mu + \frac{1}{2}\sigma^2)t \\ r &= \mu + \frac{1}{2}\sigma^2 \end{aligned}$$

Note that the expected simply compounded rate of return on the asset in this lemma is  $e^{\mu + \frac{1}{2}\sigma^2} - 1$ , and the simply compounded risk-free rate of return is  $e^r - 1$ . Thus, the lemma says that all assets whose values follow random walks in a risk-neutral universe earn the same expected simply compounded rate of return as the risk-free asset. When investors are risk-neutral, they don’t demand a risk premium, so the expected return on a risky asset has no premium above the risk-free rate of return.

**Theorem 1** *In a risk-neutral universe, if the value of an asset follows the random walk  $\frac{ds}{s} = e^{\mu dt + \sigma dX} - 1$  where  $dX$  is  $N[0, dt]$ , with  $S = s(0)$  = the asset value at time  $t$ , and if  $C$  is the value at time  $t = 0$  of a European call option on the asset with strike price  $E$  and time to expiration  $t$ , and if  $r$  is the continuously compounded risk-free rate, then:*

$$C = SN \left( \frac{\log(S/E) + (r + \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} \right) - e^{-rt}EN \left( \frac{\log(S/E) + (r - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} \right)$$

where  $N$  is the normal distribution cumulative density function.

This is the same Black-Scholes equation for European call option pricing that holds in our own risk-averse universe.

**Proof:**

By Theorem 4.1 in [3]:

$$s(t) = Se^{\mu t + \sigma X} \quad \text{where } X \text{ is } N[0, t]$$

Let  $X = \sqrt{t}Y$ . Then:

$$s(t) = Se^{\mu t + \sigma\sqrt{t}Y} \quad \text{where } Y \text{ is } N[0, 1]$$

The value at expiration of the call option (at time  $t$ ) is  $\max(s(t) - E, 0)$ . Thus, because the universe is risk-neutral, by Definition 1 above:

$$\begin{aligned} C &= e^{-rt}E(\max(s(t) - E, 0)) \\ &= e^{-rt}E\left(\max\left(Se^{\mu t + \sigma\sqrt{t}Y} - E, 0\right)\right) \\ &= e^{-rt} \int_{-\infty}^{\infty} \max\left(Se^{\mu t + \sigma\sqrt{t}y} - E, 0\right) \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \end{aligned}$$

Let:

$$z = \frac{\log(E/S) - \mu t}{\sigma\sqrt{t}}$$

Note that:

$$\max\left(Se^{\mu t + \sigma\sqrt{t}y} - E, 0\right) = \begin{cases} 0 & \text{if } y \leq z \quad (\text{the call expires out of the money}) \\ Se^{\mu t + \sigma\sqrt{t}y} - E & \text{if } y > z \quad (\text{the call expires in the money}) \end{cases}$$

Then:

$$\begin{aligned}
C &= e^{-rt} \int_{-\infty}^z \left( 0 \times \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \right) dy + \\
&\quad e^{-rt} \int_z^{\infty} \left( S e^{\mu t + \sigma \sqrt{t} y} - E \right) \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \\
&= e^{-rt} S \int_z^{\infty} e^{\mu t + \sigma \sqrt{t} y} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy - e^{-rt} E \int_z^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \\
&= e^{-rt} S \int_z^{\infty} \frac{1}{\sqrt{2\pi}} e^{\mu t + \sigma \sqrt{t} y - y^2/2} dy - e^{-rt} E [1 - N(z)] \\
&= (\text{complete the square}) \\
&\quad e^{-rt} S \int_z^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(y - \sigma \sqrt{t})^2/2 + (\mu + \frac{1}{2}\sigma^2)t} dy - e^{-rt} E [1 - N(z)] \\
&= e^{-rt + (\mu + \frac{1}{2}\sigma^2)t} S \int_z^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(y - \sigma \sqrt{t})^2/2} dy - e^{-rt} E [1 - N(z)] \\
&= (\text{substitute } x = y - \sigma \sqrt{t}) \\
&\quad e^{-rt + (\mu + \frac{1}{2}\sigma^2)t} S \int_{z - \sigma \sqrt{t}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx - e^{-rt} E [1 - N(z)] \\
&= e^{-rt + (\mu + \frac{1}{2}\sigma^2)t} S \left[ 1 - N(z - \sigma \sqrt{t}) \right] - e^{-rt} E [1 - N(z)] \\
&= (\text{because } 1 - N(w) = N(-w)) \\
&\quad e^{-rt + (\mu + \frac{1}{2}\sigma^2)t} S N(-z + \sigma \sqrt{t}) - e^{-rt} E N(-z) \\
&= e^{-rt + (\mu + \frac{1}{2}\sigma^2)t} S N \left( -\frac{\log(E/S) - \mu t}{\sigma \sqrt{t}} + \sigma \sqrt{t} \right) - e^{-rt} E N \left( -\frac{\log(E/S) - \mu t}{\sigma \sqrt{t}} \right) \\
&= (\text{because } -\log(E/S) = \log(S/E)) \\
&\quad e^{-rt + (\mu + \frac{1}{2}\sigma^2)t} S N \left( \frac{\log(S/E) + \mu t}{\sigma \sqrt{t}} + \sigma \sqrt{t} \right) - e^{-rt} E N \left( \frac{\log(S/E) + \mu t}{\sigma \sqrt{t}} \right) \\
&= e^{-rt + (\mu + \frac{1}{2}\sigma^2)t} S N \left( \frac{\log(S/E) + (\mu + \frac{1}{2}\sigma^2 + \frac{1}{2}\sigma^2)t}{\sigma \sqrt{t}} \right) - \\
&\quad e^{-rt} E N \left( \frac{\log(S/E) + (\mu + \frac{1}{2}\sigma^2 - \frac{1}{2}\sigma^2)t}{\sigma \sqrt{t}} \right) \\
&= (\text{by Lemma 1}) \\
&\quad S N \left( \frac{\log(S/E) + (r + \frac{1}{2}\sigma^2)t}{\sigma \sqrt{t}} \right) - e^{-rt} E N \left( \frac{\log(S/E) + (r - \frac{1}{2}\sigma^2)t}{\sigma \sqrt{t}} \right)
\end{aligned}$$

## References

- [1] John Norstad. The normal and lognormal distributions.  
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