SUMSRI NUMBER THEORY SEMINAR
HOMEWORK ASSIGNMENT FOR TUESDAY, JUNE 17

Consider abelian groups $(G, \circ)$, $(\Gamma, *)$, and $(G', \star)$ with identities e, ε, and e', respectively. Recall that a *group homomorphism* is a well-defined map $\phi : G \to \Gamma$ such that

$$\phi(g \circ h) = \phi(g) * \phi(h) \qquad \text{for all } g,\, h \in G.$$

Let $\phi : G \to \Gamma$ and $\hat{\phi} : \Gamma \to G'$ be two such homomorphisms. We say that the composition

$$G \xrightarrow{\ \phi\ } \Gamma \xrightarrow{\ \hat{\phi}\ } G'$$

is *exact* if $\phi(G) = \ker(\hat{\phi})$ as subsets of Γ.

a. For subgroups $H \subseteq G$ and $\Lambda \subseteq \Gamma$, define the *inclusion* homomorphism $\iota : H \hookrightarrow G$ as the map $g \mapsto g$, and the *projection* homomorphism $\pi : \Gamma \twoheadrightarrow \Gamma/\Lambda$ as the map $\gamma \mapsto \gamma \pmod{\Lambda}$. Show that the compositions

$$\{e\} \xrightarrow{\ \iota\ } \ker(\phi) \xrightarrow{\ \iota\ } G \xrightarrow{\ \phi\ } \phi(G) \xrightarrow{\ \pi\ } \{\varepsilon\}$$

are exact for each pair of compositions.
Hint: Consider $H = \ker(\phi)$ and $\Lambda = \phi(G)$.

b. Recall that we call G the *domain* of ϕ and Γ the *codomain* of ϕ. Similarly, we call $\ker(\phi)$ the *kernel* of ϕ and the quotient group $\operatorname{coker}(\phi) = \Gamma/\phi(G)$ the *cokernel* of ϕ. Show that ϕ is injective if and only if it has trivial kernel; and that ϕ is surjective if and only if it has trivial cokernel.

c. Denote $G[\phi] = \ker(\phi)$ as the kernel of ϕ. Show that the following sequence is exact:

$$\{e\} \xrightarrow{\ \iota\ } G[\phi] \xrightarrow{\ \iota\ } G \xrightarrow{\ \phi\ } \Gamma \xrightarrow{\ \pi\ } \frac{\Gamma}{\phi(G)} \xrightarrow{\ \pi\ } \{\varepsilon\}$$

That is, show that each pair compositions is exact.
Hint: Consider the First Isomorphism Theorem.

d. Show that $\phi(G) \cap \Gamma[\hat{\phi}] = \phi\big(G[\hat{\phi} \circ \phi]\big)$.

e. Denote the quotient groups $\overline{\Gamma} = \Gamma/\phi(G)$ and $\overline{G'} = G'/\big(\hat{\phi} \circ \phi\big)(G)$. Show that the map $\hat{\phi} : \overline{\Gamma} \to \overline{G'}$ which sends

$$\gamma \mod \phi(G) \quad \mapsto \quad \hat{\phi}(\gamma) \mod \big(\hat{\phi} \circ \phi\big)(G)$$

is a well-defined group homomorphism.

f. Show that the following sequence is exact:

$$\{\overline{\varepsilon}\} \xrightarrow{\ \iota\ } \frac{\Gamma[\hat{\phi}]}{\phi\big(G[\hat{\phi} \circ \phi]\big)} \xrightarrow{\ \iota\ } \frac{\Gamma}{\phi(G)} \xrightarrow{\ \hat{\phi}\ } \frac{G'}{\big(\hat{\phi} \circ \phi\big)(G)} \xrightarrow{\ \pi\ } \frac{G'}{\hat{\phi}(\Gamma)} \xrightarrow{\ \pi\ } \{\overline{e'}\}$$

Hint: Consider part (c) for the group homomorphism $\hat{\phi} : \overline{\Gamma} \to \overline{G'}$. Use part (d) to help compute the kernel and cokernel of this map.