TRIANGULAR TOEPLITZ MATRICES

KENNETH R. DRIESSEL

Date: November 24, 2006.

In this section we discuss matrix equations having the form $Ax = b$ where A is a lower triangular Toeplitz matrix. We begin by showing that the division algorithm for univariate polynomials leads to such equations.

A matrix is "Toeplitz" if has constant diagonals.

Definition: Let T be an $n \times n$ matrix. Then T is a **Toeplitz matrix** if, for all i, j, k, l, between 1 and n, $j - i = l - k$ implies $T(i, j) = T(k, l)$.

Example: For $n = 3$, Toepliz matrices have the following form:

$$\begin{pmatrix} t_0 & t_{-1} & t_{-2} \\ t_1 & t_0 & t_{-1} \\ t_2 & t_1 & t_0 \end{pmatrix}.$$

Let

- $Q := a_q X^q + a_{q-1} X^{q-1} + \cdots + a_0$ and
- $P := b_p X^p + b_{p-1} X^{p-1} + \cdots + b_0$.

Recall that there exist unique polynomials U (the quotient) and R (the remainder) such that $P = QU + R$ and the degree of R is les than the degree of Q. We can get U and R by first solving for U by matching coefficients and the setting $R := P - QU$. We assume that $p \geq q$. (The other case is trivial.) Let

- $U = u_m X^m + u_{m-1} X^{m-1} + \cdots + u_0$

where m is defined by $q + m := p$. We have

- $QU = a_q u_m X^{q+m} + (a_{q-1} u_m + a_q u_{m-1}) X^{q+m-1} + \cdots + a_0 u_0$.

Equating coefficients we obtain the following matrix equation for the u's:

$$\begin{pmatrix} a_q & 0 & 0 & \dots & 0 & 0 \\ a_{q-1} & a_q & 0 & \dots & 0 & 0 \\ a_{q-2} & a_{q-1} & a_q & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{q-m+1} & a_{q-m+2} & a_{q-m+3} & \dots & a_q & 0 \\ a_{q-m} & a_{q-m+1} & a_{q-m+2} & \dots & a_{q-1} & a_q \end{pmatrix} \begin{pmatrix} u_m \\ u_{m-1} \\ u_{m-2} \\ \vdots \\ u_1 \\ u_0 \end{pmatrix} = \begin{pmatrix} b_p \\ b_{p-1} \\ b_{p-2} \\ \vdots \\ b_{p-m+1} \\ b_{p-m} \end{pmatrix}.$$

A lower triangular Toeplitz can be ill-conditioned. The condition of the lower triangular Toeplitz matrix which appears above determines the condition of the division algorithm. In the euclidean algorithm for polynomials, a number of such matrices appear; they determine the condition of this algorithm.

Definition: Let $\mathbf{t} := (t_0, t_1, \dots, t_{n-1})$. We shall use LToep($\mathbf{t}$) to denote the lower triangular Toeplitz matrix which has $\mathbf{t}$ as its first column. In other words, $\mathrm{LToep}(\mathbf{t})(i, j) := t_{i-j}$ if $j \leq i$ and $:= 0$ otherwise.

Example: For example, when $n = 3$, we have

$$\mathrm{LToep}(t_0, t_1, t_2) = \begin{pmatrix} t_0 & 0 & 0 \\ t_1 & t_0 & 0 \\ t_2 & t_1 & t_0 \end{pmatrix}.$$

Definition: We shall use Z_n (or simply Z) to denote the **lower shift matrix** which is defined by $Z_n := \text{LToep}(0, 1, 0, \ldots, 0)$. In other words, Z_n is the lower triangular Toeplitz matrix which is zero except for ones on the first sub-diagonal.

Example: For example, when $n = 3$, we have

$$Z = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The following proposition explains the use of the word "shift" in the last definition. The proposition says that multiplication by Z on the left shifts the rows of a matrix down and multiplication by Z on the right shifts the columns to the left.

Proposition 1. *Let A be a $n \times n$ matrix. Then*

- $(ZA)(i, j) = A(i - 1, j)$ *if* $i > 1$ *and* $= 0$ *otherwise.*
- $(AZ)(i, j) = A(i, j + 1)$ *if* $j < n$ *and* $= 0$ *otherwise.*

Proposition 2. *The lower shift matrix is nilpotent. In fact, Z_n^k is the lower triangular Toeplitz matrix which is* 0 *except for ones on the* $(k + 1)$*st subdiagonal. In particular,* $Z_n^n = 0$.

Example: For example, when $n = 3$, we have

$$Z^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Proposition 3. *Every lower triangular Toeplits matrix can be represented as a linear combination of powers of the lower shift matrix:*

$$LToep(t_0, \ldots, t_{n-1}) = t_0 I + t_1 Z_n + t_2 Z_n^2 + \cdots + t_{n-1} Z_n^{n-1}$$

Proposition 4. *The $n \times n$ lower triangular matrices with entries from a commutative ring form a commutative ring.*

Example: For example, when $n = 3$, we have

$$\begin{aligned} \text{LToep}(t_0, t_1, t_2)\text{LToep}(u_0, u_1, u_2) &= (t_0 I + t_1 Z + t_2 Z^2)(u_0 I + u_1 Z + u_2 Z^2) \\ &= t_0 u_0 I + (t_1 u_0 + t_0 u_1) Z + (t_2 u_0 + t_1 u_1 + t_0 u_2) Z^2. \end{aligned}$$

Proposition 5. *If the diagonal entry of a lower triangular Toeplitz matrix is non-zero then it is non-singular. Its inverse is a lower triangular Toeplitz matrix.*

Proof. (Sketch.) (Joel Roberts suggested this proof.) Let L be any strictly lower triangular Toeplitz matrix; in other words, there exist $t_0, t_1, \ldots, t_{n-1}$ such than $L = \text{LToep}(0, t_1, \ldots, t_{n-1})$. Then L is nilpotent, in particular, $L^n = 0$. We then have $(I - L)^{-1} = I + L + L^2 + \cdots + L^{n-1}$. □