# SIMPLY EXISTENTIAL FORMULAS

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I mainly follow Tarski(1951,1967) .

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Recall the following results.

**Proposition 1. Root counting.** *(Sturm, 1835) Let $R$ be a real closed field. Let $P$ be a polynomial in $R[X]$. Let $a < b$ be elements of $R \cup \{-\infty, \infty\}$ which are not roots of $P$. Let $S(P, P')$ be the signed remainder sequence determined by $P$ and its derivative $P'$. Then*

$$\#\{r \in R : a < r < b \wedge P(r) = 0\} = V(S(P, P'); a) - V(S(P, P'); b).$$

**Proposition 2. Root counting** *(Sylvester,Tarski). Let $R$ be a real closed field. Let $P$ and $Q$ be polynomials in $R[X]$. Let $a < b$ be elements of $R \cup \{-\infty, \infty\}$ which are not roots of $P$ or $Q$. Let $S(P, P'Q)$ be the signed remainder sequence determined by polynomials $P$ and $P'Q$. Then*

$$\begin{aligned} &\#\{r \in R : a < r < b \wedge P(r) = 0 \wedge Q(r) > 0\} - \\ &\#\{r \in R : a < r < b \wedge P(r) = 0 \wedge Q(r) < 0\} \\ &\quad = V(S(P, P'Q); a) - V(S(P, P'Q); b). \end{aligned}$$

Let $P(X)$ be a polynomial in $R[X]$ where $R$ is a real closed field. Note that (by Sturm's theorem) the following two formulas are equivalent:

- $(\exists X)P(X) = 0$
- $V(S(P, P'); -\infty) > V(S(P, P'); \infty)$.

In other words, simply existential formulas which assert the existence of a root of a polynomial are equivalent to quantifier free formulas.

**Example:** Consider the polynomial $f = 8 - 4X + 6X^2 - 3X^3 - 2X^4 + X^5 = (X^2 + 1)(X - 2)^2(X + 2)$. The Sturm sequence is as follows:

$$f_0 = 8 - 4X + 6X^2 - 3X^3 - 2X^4 + X^5$$

$$f_1 = -4 + 12X - 9X^2 - 8X^3 + 5X^4$$

$$f_2 = -\frac{192}{25} + \frac{56}{25}X - \frac{72}{25}X^2 + \frac{46}{25}X^3$$

$$f_3 = \frac{2500}{529} - \frac{17500}{529}X + \frac{8125}{529}X^2$$

$$f_4 = \frac{33856}{4225} - \frac{16928}{4225}X.$$

We evaluate the polynomials $f_i(X)$ at $x = \pm\infty$ to get the following table of signs which determine variations $V(x)$:

| $x$ | $f_0(x)$ | $f_1(x)$ | $f_2(x)$ | $f_3(x)$ | $f_4(x)$ | $V(x)$ |
|---|---|---|---|---|---|---|
| $--$ | $--$ | $--$ | $--$ | $--$ | $--$ | $--$ |
| $-\infty$ | $-1$ | $+1$ | $-1$ | $+1$ | $+1$ | 3 |
| $+\infty$ | $+1$ | $+1$ | $+1$ | $+1$ | $-1$ | 1 |

Hence the formula $(\exists X)f(X) = 0$ is equivalent to the formula $3 > 1$.

Now consider the formula:

$$(\exists X)(P(X) = 0 \wedge Q(X) > 0).$$

We can use Tarski's generalization of Sturm's theorem to handle this case. In particular, we can use Sturm's theorem to determine the number $n$ of roots of $P$. We can use Sturm's theorem again to determine the number $n_0$ of roots common to $P$ and $Q$. (In particular, we count the roots of $P^2+Q^2$.) Let $n_+$ denote the number of roots of $P$ at which $Q$ is positive and let $n_-$ be the number of such roots at which $Q$ is negative. We can use Tarski's generalization to compute the value of $n_+ - n_-$. This quantity together with the relation $n_+ + n_- = n - n_0$ determines $n_+$ and $n_-$. In this way we get a quantifieer free formula which is equivalent to the simply existential formula given above.

We also want to consider some other simply existential formulas. In particular, we want to see that we can apply similar analyses in more general situations.

Consider the following formula:

$$(\exists X)(P(X) = 0 \wedge Q_1(X) > 0 \wedge Q_2(X) > 0.$$

Assume that the number of roots of $P$ is finite. Define the following quantities associated with the given polynomials:

$$\begin{aligned}
m_1 &:= \#\{x \in R : P(x) = 0 \wedge Q_1(x) > 0 \wedge Q_2(x) > 0\}\\
m_2 &:= \#\{x \in R : P(x) = 0 \wedge Q_1(x) > 0 \wedge Q_2(x) < 0\}\\
m_3 &:= \#\{x \in R : P(x) = 0 \wedge Q_1(x) < 0 \wedge Q_2(x) > 0\}\\
m_4 &:= \#\{x \in R : P(x) = 0 \wedge Q_1(x) < 0 \wedge Q_2(x) < 0\}\\
n_1 &:= \#\{x \in R : P(x) = 0 \wedge Q_1(x)^2 Q_2(x)^2 > 0\}\\
n_2 &:= \#\{x \in R : P(x) = 0 \wedge Q_1(x) Q_2(x)^2 > 0\}\\
n_3 &:= \#\{x \in R : P(x) = 0 \wedge Q_1(x)^2 Q_2(x) > 0\}\\
n_4 &:= \#\{x \in R : P(x) = 0 \wedge Q_1(x) Q_2(x) > 0\}.
\end{aligned}$$

Note that we have the following relations between these quantities:

$$\begin{aligned}
m_1 + m_2 + m_3 + m_4 &= n_1\\
m_1 + m_2 &= n_2\\
m_1 + m_3 &= n_3\\
m_1 + m_4 &= n_4.
\end{aligned}$$

In other words, we have

$$\begin{pmatrix} 1 & 1 & 1 & 1\\ 1 & 1 & 0 & 0\\ 1 & 0 & 1 & 0\\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} m_1\\ m_2\\ m_3\\ m_4 \end{pmatrix} = \begin{pmatrix} n_1\\ n_2\\ n_3\\ n_4 \end{pmatrix}.$$

Note that the matrix is invertible. In particular, we have

$$m_1 = \frac{1}{2}(-n_1 + n_2 + n_3 + n_4).$$

Note that, instead of counting the number of roots of $P$ which satisfy two inequalities $Q_1(X) >$ and $Q_2(X) > 0$, we can separately count the number of roots of $P$ which satisfy only one inequality. In this way we have reduced the problem involving two inequalities to the problem involving only one inequality which we already know how to handle. Clearly, we can generalize this analysis to handle formulas having the following form:

$$(\exists X)(P(X) = 0 \wedge Q_1(X) > 0 \wedge \cdots \wedge Q_l(X) > 0).$$

We now consider the formula

$$(\exists X)(Q(X) > 0)$$

where $Q(X) := c_q X^q + \cdots + c_0$. Let $\mathrm{Pos}(Q, Y^-)$ denote the following formula:

$$Q(Y) = 0 \wedge (\exists \epsilon)(\forall Z)((Y - \epsilon < Z \wedge Z < Y) \rightarrow Q(Z) > 0).$$

This formula says that $Q$ vanishes at $Y$ and is positive just to the left of $Y$. Note that this formula is equivalent to the following quantifier free formula:

$$\begin{aligned} Q(Y) = 0 \wedge ((&-Q'(Y) > 0) \\ &\vee (Q'(Y) = 0 \wedge (-1)^2 Q''(Y) > 0) \\ &\vee \ldots \\ &\vee (Q'(Y) = 0 \wedge \cdots \wedge (-1)^q Q^{(q)}(Y) > 0)). \end{aligned}$$

It is clear that the following two formulas are equivalent:

- $(\exists X)(Q(X) > 0)$
- $c_q > 0 \vee (\exists X)\mathrm{Pos}(Q, X^-)$.

In this way we have reduced a simply existential formula involving an inequality to a simply existential formula involving the disjunction of a quantifier free formula and a simply existential formula which is in a form that we already know how to handle.

Clearly we can generalize this analysis to handle formulas having the following form:

$$(\exists X)(Q_1(X) > 0 \wedge \cdots \wedge Q_l(X) > 0).$$