REAL ALGEBRAIC GEOMETRY

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Real Rings

References: In this section I mainly follow Chapter 4 “Real Rings” in Prestel and Delzell(2001) .

Notation: Throughout this section, I use the following notation:

- A denotes a commutative ring with identity 1.
- A^2 denotes the set of squares of elements of A: in symbols,
$$A^2 := \{a^2 : a \in A\}.$$
- $\sum A^2$ denotes the set of sums of squares of elements of A; in symbols,
$$\sum A^2 := \{a_1^2 + \cdots + a_m^2 : a_i \in A\}$$
.
- $A^{\times n}$ denotes the nth cartesian power of A. (Prestel and Delzell use $A^{(n)}$ to denote this cartesian power.)
- For subsets U and V of A,
$$\begin{aligned} -U &:= \{-u : u \in U\}, \\ U + V &:= \{u + v : u \in U, v \in V\}, \\ UV &:= U \cdot V := \{uv : u \in U, v \in V\}. \end{aligned}$$

We shall look at the possible “orderings” of A.

Definition: A subset T of A is a **pre-positive cone** or **pre-ordering** of A if it satisfies the following conditions:

- $A^2 \subseteq T$
- $T + T \subseteq T$
- $T \cdot T \subseteq T$
- $-1 \notin T$.

Remark: In elementary geometry a cone C with its vertex at the origin has the following property: If p is a point on the cone C which is different than the origin, then every point on the half-line from the vertex thru the point p is also on the cone. More generally, in analytical geometry, a subset C of $\mathbb{R}^k$ is a “cone” if it is closed under multiplication by weakly positive (nonnegative) scalars. Note that a pre-positive cone T satisfies $A^2T \subseteq T$.

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In other words, this use of the word "cone" generalizes its use in analytical geometry.

Proposition 1. *The intersection of a nonempty family of pre-positive cones on A is a prepositive cone on A.*

Proof. The proof is straightforward. I leave the details to the reader. ☐

Proposition 2. Smallest pre-positive cone. *Let $T := \sum A^2$. Then T satisfies:*

- $0 \in T$ *and* $1 \in T$.
- $A^2 \subseteq T$.
- $T + T \subseteq T$.
- $T \cdot T \subseteq T$.

If $-1 \notin \sum A^2$ then $\sum A^2$ is the smallest pre-positive cone on A.

Proof. The proofs are straight-forward. For example, $0 = 0^2$ and $1 = 1^2$ and hence 0 and 1 are in T. Also note that the product of sums of squares is a sum of squares:

$$(\sum_{i=1}^{m} a_i^2)(\sum_{j=1}^{n} b_j^2 = \sum_{i=1}^{m}\sum_{j=1}^{n} a_i^2 b_j^2.$$

☐

Definition: The ring A is **semi-real** if -1 is not the sum of squares in A; in symbols, $-1 \notin \sum A^2$.

Proposition 3. *If A is semi-real then $\sum A^2$ is a pre-positive cone on A.*

Here are some examples.

Example: Let A be the ring of integers $\mathbb{Z}$ and let T be the set of natural numbers $\mathbb{N}$. Note that $\mathbb{N} = \sum \mathbb{Z}^2$.

Example: Let A be a subring of the real numbers $\mathbb{R}$. Let $T := \{a \in A : 0 \leq a\}$. It is easy to see that T is a pre-positive cone for A.

Example: Let $A := \mathbb{R}[X_1, \dots, X_k]$ and let $T := \sum A^2$.

Example: Let X be a topological space and let A be the ring of continuous real-valued functions on X; in symbols, $A := C(X \to \mathbb{R})$. Addition and multiplication are defined point-wise as usual. Let T be the set of weakly positive (nonnegative) functions; in symbols, $T := \{f \in A : (\forall x \in X)\ f(x) \geq 0\}$. It is easy to see that T is a pre-positive cone on A.

Example: Let A be the set of continuous real-valued functions on $\mathbb{R}$; in symbols, $A := C(\mathbb{R} \to \mathbb{R})$. Addition and multiplication are defined point-wise as usual. Let $T := \{f \in A : f(0) \geq 0\}$. It is easy to see that T is a pre-positive cone on A.

Example: Let $A := \mathbb{R}[X]$ be the ring of polynomials in a single variable X. Addition and multiplication are defined as usual for polynomials. Let

$T := \{f \in A : x \geq 0 \implies f(x) \geq 0\}$. Note that the polynomial X is in T but it is not a sum of squares.

Remark: Let T be a pre-positive cone on A. Define the binary relation $x \leq y$ on A by $y - x \in T$. Then it is easy to see that $\leq$ is reflexive ($x \leq x$) and transitive ($x \leq y \wedge y \leq z \implies x \leq z$). The last example shows that this binary relation need not be anti-symmetric; that is, the following implication need not hold: $x \leq y \wedge y \leq x \implies x = y$.

Definition: Let T be a pre-positive cone on A. Then T is **total** if $T \cup -T = A$.

Note that if T is a total pre-positive cone on A then the binary relation defined above has the following property: $\forall x, y \in A, x \leq y \vee y \leq x$.

Proposition 4. *Let T be a total, pre-positive cone on A. Then $T \cap -T$ is an ideal of A.*

Proof. Let $I := T \cap -T$.

Claim: The following conditions are equivalent: for all $a \in A$

- $a \in I$.
- $-a \in I$.
- $a \in I \wedge -a \in I$.
- $a \in T \wedge -a \in T$.

It is easy to check this claim. I leave the details to the reader.

Claim: $I + I \subset I$.

We have

$$\begin{aligned} a, b \in I &\implies a, b, -a, -b \in T \\ &\implies a + b \in T \wedge -(a+b) = (-a) + (-b) \in T \\ &\implies a + b \in T. \end{aligned}$$

Claim: $TI \subseteq I$.

We have

$$\begin{aligned} t \in T \wedge a \in I &\implies t \in T \wedge a \in T \wedge -a \in T \\ &\implies ta \in T \wedge -ta = t(-a) \in T \\ &\implies ta \in I. \end{aligned}$$

Claim: $-TI \subseteq I$.

We have

$$\begin{aligned} t \in -T \wedge a \in T \wedge -a \in T &\implies -t \in T \wedge a \in T \wedge -a \in T \\ &\implies -(ta) \in T \wedge ta = (-t)(-a) \in T \\ &\implies ta \in I. \end{aligned}$$

Claim: $AI \subseteq I$.

Here we use $T \cup -T = A$. We have

$$\begin{aligned} x \in A \wedge a \in I &\implies (x \in T \vee -x \in T) \wedge a \in I \\ &\implies (x \in T \wedge a \in I) \vee (-x \in T \wedge a \in I) \\ &\implies xa \in I \vee -xa \in I. \end{aligned}$$

□

Definition: For any subset T of A, the subset $\mathrm{Supp}(T) := T \cap -T$ is the **support** of T.

Definition: A subset T of A is a **positive cone** of A if the following conditions are satisfied:

- T is a pre-positive cone,
- T is total, and
- The support of T is a prime ideal of A.

Definition: Recall that an ideal I in A is a **prime ideal** if I is proper (that is, $I \neq A$) and satisfies the following condition for all a, b in A: $ab \in I \implies a \in I \vee b \in I$.

The following proposition shows that last item of the definition is irrelevant for fields.

Proposition 5. *Assume that A is a field. Let T be a pre-positive cone for A. Then Supp$(T) = 0$.*

Proof. Suppose that $\mathrm{Supp}(T) \neq 0$. Then there exists an element x in $\mathrm{Supp}(T)$ which is different than 0. Note $-(1/x) = -x(1/x)^2 \in T$. Then $-1 = -(1/x)x \in T$. □

Let x be an element of A and let T be a subset of A which satisfies $0 \in T$ and $1 \in T$. Then $T \subseteq T + xT$ and $x \in T + xT$. The following extension lemma shows that we can ofteh extend pre-positive cones in this way.

Proposition 6. Extension lemma. *Let T be a pre-positive cone on A. Then for every pair of elements a and b in A, if $ab \in -T$ then either $T + aT$ or $T + bT$ is a pre-positive cone.*

Proof. Claim: For every element x of A, the set $U := T + xT$ satisfies $A^2 \subseteq U, U + U \subseteq U$ and $U \cdot U \subseteq U$.

The proof of this claim is straight-forward. I leave the details to the reader.

To finish the proof we need to see that -1 is not in $T + aT$ or -1 is not in $T + bT$. The proof of the following claim will accomplish this.

Claim: If $-1 \in T + aT, -1 \in T + bT$ and $-ab \in T$ then $-1 \in T$.

Suppose that $-1 = p_1 + aq_1$ and $-1 = p_2 + bq_2$ for elements p_1, q_1, p_2, q_2 in T. It follows that $1 + p_1 = -aq_1$ and $1 + p2 = -aq_2$ and hence

$$1 + p_1 + p_2 + p_1p_2 = (1 + p_1)(1 + p_2) = abq_1q_2.$$

From this equation we get $-1 = p_1 + p_2 + p_1p_2 + (-ab)q_1q_2$ is an element of T. □

Proposition 7. *If a pre-positive cone is maximal then it is a positive cone.*

Proof. Let P be a maximal cone.

Claim: $A = P \cup -P$.

Consider any a in A. Note that $a(-a) \in -T$. By the extension lemma we then have that either $P + aP$ or $P + (-a)P$ is a prepositive cone. By maximality of P we then have either $P + aP = P$ or $P + (-a)P = P$. Hence either $a \in P$ or $-a \in P$.

Claim: The set $I := \mathrm{Supp}(P) := P \cap -P$ is a prime ideal.

Note that I is an ideal since P is total. We only need to show that I is prime, that is, $ab \in I \implies a \in I \vee b \in I$. Suppose $ab \in I$ and $a \notin I$. Then $a \notin P$ or $a \notin -P$.

Case: $a \notin P$

Then $P + aP$ is not a pre-positive cone (since P is maximal). Since $ab \in I \subseteq -P$, by the extension lemma, we have $P = P + bP$. Since $a(-b) \in -P$, by the extension lemma, we have $P = P + (-b)P$. Thus $b \in P$ and $-b \in P$ and hence $b \in I$.

Case: $-a \notin P$.

Then $P + (-a)P$ is not pre-positive. Since $(-a)(-b) \in -P$ we have $P + (-b)P = P$ and since $(-a)b \in -P$ we have $P + bP = P$. Thus $b \in P$ and $-b \in P$ and again $b \in I$. □

Proposition 8. *Every prepositive cone may be extended to a positive cone.*

Proof. Let T be a prepositive cone. Note that the union of any chain of pre-positive cones containing T is a pre-positive cone containing T. We then use Zorn's lemma to get the existence of a maximal pre-positive cone P containing T. By the last result P is a positive cone. □

Example: (Steve Kaliszewski (Arizona State University) suggested this example to me.) Let $A := \mathbb{R}[X]$ be the algebra of polynomials in one variable X. For r in $\mathbb{R}$, let $T_r := \{p \in A : (\forall x \geq r)\ p(x) \geq 0\}$. Note that each of these sets is a pre-positive cone. Also note that if $r < s$ then T_r is a proper subset of T_s. Consequently, this algebra has infinite ascending chains of pre-positive cones and infinite descending chains of pre-positive cones.