REAL ALGEBRAIC GEOMETRY

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Quadratic Forms

References: In this section I mainly follow the book Basu, Pollack and Roy(2003) .

Notation: Throughout this section K denotes an ordered field.

In this section we review a few basic properties of quadratic forms.

Definition: Let U, V and W be vector spaces over K. Then a map $\Psi : U \times V \to W$ is **bilinear** if it is linear in each of its arguments:

- $\Psi(au, v) = a\Psi(u, v)$
- $\Psi(u_1 + u_2, v) = \Psi(u_1, v) + \Psi(u_2, v)$
- $\Psi(u, av) = a\Psi(u, v)$
- $\Psi(u, v_1 + v_2) = \Psi(u, v_1) + \Psi(u, v_2)$.

A bilinear map $\Psi : V \times V \to W$ is **symmetric** if

- $\Psi(u, v) = \Psi(v, u)$.

If the bilinear map takes its values in the field K, that is, $\Psi : U \times V \to K$, then it is called a **bilinear form**. A function $\Phi : V \to K$ is a **quadratic form** if there is a symmetric bilinear form $\Psi : V \times V \to K$ such that $\Phi(v) = \Psi(v, v)$.

The following identity (which is called the **polarization identity**) indicates that the symmetric bilinear form can be recovered from the quadratic form:

$$\Psi(u + v, u + v) = \Psi(u, u) + \Psi(u, v) + \Psi(v, u) + \Psi(v, v).$$

If we have an ordered basis $v_1, \ldots, v_n$ for V then V is isomorphic to K^n; in particular, we can map $v = f_1v_1 + \cdots + f_nv_n$ to the column vector $f = (f_1, \ldots, f_n)^T$. We can also use the basis to associate a symmetric matrix with a quadratic form Φ on V; in particular, we set $M_{i,j} := \Phi(v_i, v_j)$. Then

$$\Phi(v) = \sum\{M_{i,j}f_if_j : i, j = 1, \ldots, n\} = f^TMf = \langle f, Mf\rangle$$

where $\langle f, g\rangle := f^Tg$ is the standard bilinear form on K^n. If G is a nonsingular matrix and $G^Tg = f$ then we have

$$\Phi(v) = \langle G^Tg, MG^Tg\rangle = \langle g, GMG^Tg\rangle.$$

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In other words, a change of basis transforms M to GMG^T. So the appropriate group action, in this setting, is the following one:

$$* := \mathrm{Gl}(n, K) \times K^{n\times n} \to K^{n\times n} : (G, M) \mapsto GMG^T$$

where $\mathrm{Gl}(n, K)$ denotes the general linear group of invertible $n \times n$ matrices with entries from K. In particular, note that $(GH) * M = G * (H * M)$.

Definition: The **rank** of a quadratic form is defined to be the rank of its associated matrix with respect to some basis. (It is easy to show that this definition does not depend on the choice of basis.)

Theorem 1. Sylvester's law of inertia. *Let $\Phi : V \to K$ be a quadratic form with rank r. Then there exist linearly independent linear forms $L_i : V \to K$ and nonzero scalars d_i, for $i = 1, \ldots, r$, such that*

$$\Phi(v) = \sum\{d_i L_i(v)^2 : i = 1, \ldots, r\}.$$

Furthermore, the following integer is a well-defined quantity:

$$\#\{d_i : d_i > 0\} - \#\{d_i : d_i < 0\}.$$

Definition: The well-defined integer of the last result is called the **signature** of the quadratic form.

Here is a matrix version of the last result: Let M be an $n \times n$ symmetric matrix. Then there is a matrix G in $\mathrm{Gl}(n, K)$ and a diagonal matrix D such that $M = GDG^T$. If r_+ is the number of diagonal entries of D which are positive and r_- is the number which are negative then $\mathrm{Rank}(M) = r_+ + r_-$ and $\mathrm{Signature}(M) = r_+ - r_-$. If $D = \mathrm{diag}(d_1, \ldots, d_n)$ and $g_1, \ldots, g_n$ are the columns of G, then

$$M = d_1 g_1 g_1^T + \cdots + d_n g_n g_n^T.$$

(This as called an "outer product" expansion of M.) In particular, note that the $g_k g_k^T$ are matrices (with rank one). Also note, for f in K^n,

$$\langle f, Mf\rangle = d_1 f^T g_1 g_1^T f + \cdots + d_n f^T g_n g_n^T f = d_1 \langle g_1, f\rangle^2 + \ldots d_n \langle g_n, f\rangle^2.$$

This displays the quadratic form as a sum of squares of linear forms. Such a representation as a sum of squares is obviously not unique. (However, if the $|d_i|$ have square roots in K then the d_i can be taken from the three values $1, -1, 0$.)

The diagonalizing matrix G is easily computed using a modified version of Gaussian elimination: If a column has a nonzero entry off the diagonal and the diagonal entry in that column is nonzero then we can use that diagonal entry as a pivot; in particular, we then use row operations to eliminate the nonzero off-diagonal entries in that column. To maintain symmetry of the matrix we also perform the same operations on the columns. There is one further modification; sometimes there is a nonzero off-diagonal entry and no nonzero pivot available to use to eliminate that entry; in this case the nonzero entry can be moved to the diagonal; here is an example of this special situation:

$$\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}^T \begin{pmatrix}0 & a\\ a & 0\end{pmatrix} \begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix} = \begin{pmatrix}2a & 0\\ 0 & -2a\end{pmatrix}.$$

For more details, see, for example, Birkhoff and Maclane(1953) .

Definition: The triple (r_+, r_-, z) of numbers, where $z := n - (r_+ + r_-)$ is the dimension of the null space (or kernel) of M, is called the **inertia** of the matrix M.

The proof of Sylvester's law of inertia is well-known. We shall not repeat the proof here. (See, for example, Birkhoff and Maclane(1953) or Basu, Pollack and Roy(2003) for proofs.)

Theorem 2. Spectral. *Let R be a real closed field. Let M be a symmetric matrix with entries from R. Then the eigenvalues of M are in R and there is an orthonormal basis of eigenvectors of M with coordinates in R.*

We need more information about real closed fields to prove this result. We do not include a proof here. We only want to note that r_+ is the number of positive eigenvalues of M and r_- is the number of negative eigenvalues of M. Hence the signature of M is the difference between the number of positive and number of negative eigenvalues of M. Since there is no obvious intrinsic inner product in this setting, this viewpoint is somewhat unnatural.