

# ORDERED RINGS

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I mainly follow MacLane and Birkhoff(1979) . The omitted proofs should be regarded as exercises.

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We begin by considering mathematical structures with the following type:

$$\mathcal{A} = (A, +, 0, \cdot, 1, -1)$$

where  $A$  is a set,  $+$  and  $\cdot$  are binary operations on  $A$  and  $0, 1, -1$  are elements of  $A$ .

**Definition:** Such a structure is a **ring** if the following conditions are satisfied:

- **associativity for addition:**  $(x + y) + z = x + (y + z)$
- **identity for addition:**  $x + 0 = x$
- **inverse for addition:**  $x + (-1) \cdot x = 0$
- **commutativity for addition:**  $x + y = y + x$
- **associativity for multiplication:**  $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
- **identity for multiplication:**  $x \cdot 1 = x$
- **commutativity for multiplication:**  $x \cdot y = y \cdot x$
- **distributivity:**  $x \cdot (y + z) = x \cdot y + x \cdot z$
- **non-triviality:**  $0 \neq 1$ .

A ring is a **field** if the following additional condition is satisfied:

- **inverse for multiplication:**  $x \neq 0 \rightarrow \exists y (x \cdot y = 1)$ .

We also consider mathematical structures with the following type:

$$\mathcal{A} = (A, +, 0, \cdot, 1, -1, P)$$

where  $P$  is a subset of  $A$ .

**Definition:** Such a structure is an **ordered ring** if it is a ring and the following conditions are satisfied:

- The set  $P$  is non-empty.
- $a \in P \wedge b \in P \rightarrow a + b \in P \wedge a \cdot b \in P$
- **trichotomy:** for all  $a \in A$ , exactly one of the following conditions holds:

$$a \in P, a = 0, -a \in P.$$

The elements of  $P$  are called the **positive elements** of the ordered ring.

An ordered ring which is an integral domain is an **ordered domain**. (We shall see below that every ordered ring is an ordered domain.) An ordered ring which is a field is an **ordered field**.

Here are some examples of ordered rings: The integral domain  $\mathbb{Z}$  of integers with the usual order. The field  $\mathbb{Q}$  of rational numbers with the usual order. The field  $\mathbb{R}$  of real numbers with the usual order.

Here is another example. Let  $A$  be an ordered domain. Consider the ring  $A[x]$  of polynomials in the variable  $x$ . Let  $P^*$  be the set of polynomials with positive leading coefficient. Then  $A[x]$  together with positive subset  $P^*$  is an ordered ring.

**Proposition 1.** *Let  $A$  be an ordered ring. Then all squares of non-zero elements are positive; in symbols,  $a \neq 0 \rightarrow a^2 \in P$ .*

*Proof.* If  $a \in P$  then  $a^2 \in P$ . If  $-a \in P$  then  $a^2 = (-a)^2 \in P$ . □

**Proposition 2.** *Let  $A$  be an ordered ring. Then  $1 \in P$  and  $-1 \notin P$ .*

*Proof.* Simply note that  $1 = 1^2$ .  $\square$

Note that it follows from the last few propositions that the field  $\mathbb{C}$  of complex numbers cannot be ordered since in this field  $-1$  is a square.

**Definition:** A ring is a **real ring** if  $-1$  is not the sum of squares.

**Proposition 3.** *Every ordered ring is real.*

**Proposition 4.** *Every ordered ring is an integral domain.*

*Proof.* We need to see that if  $a$  and  $b$  are non-zero then their product  $a \cdot b$  is non-zero. Consider the products:  $(\pm a) \cdot (\pm b)$ . At least one of them is positive.  $\square$

**Proposition 5.** *Every ordered ring has characteristic 0.*

*Proof.* Note that every sum  $1 + 1 + \cdots + 1$  is positive.  $\square$

**Definition:** Let  $A$  and  $A'$  be ordered rings with sets of positive elements  $P$  and  $P'$  respectively. Then a ring morphism (that is, a homomorphism)  $h : A \rightarrow A'$  is an **order morphism** or **order homomorphism** if the image of every positive element of  $A$  is a positive element of  $A'$ ; in symbols,

$$h(P) := \{h(x) : x \in P\} \subseteq P'.$$

**Proposition 6.** *Every order morphism between ordered rings is a monomorphism (that is, is injective).*

*Proof.* Suppose that  $a \neq 0$ . If  $a$  is positive then  $h(a)$  is positive and hence different than 0. If  $-a$  is positive then  $(-1) \cdot h(a) = h(-a)$  is positive. Hence  $(-1) \cdot h(a) \neq 0$  and  $h(a) \neq 0$ .  $\square$

**Proposition 7.** *Let  $A$  be an ordered ring. Then there is a unique ring morphism  $\mathbb{Z} \rightarrow A$ . This ring morphism is an order morphism.*

**Proposition 8.** *Let  $A$  be an ordered ring. Then there is only one way to make the ring  $\mathbb{Z}$  of integers into an ordered ring.*

**Definition:** Let  $A$  be an ordered ring with set  $P$  of positive elements. Define the binary relation **less than** by  $a < b$  if  $b - a \in P$ . Also  $b > a$  means  $a < b$  and  $a \leq b$  means  $a < b$  or  $a = b$ . Note  $0 < b$  iff  $b \in P$ .

**Proposition 9.** *Let  $A$  be an ordered ring. Then the relation  $<$  satisfies the following conditions:*

- **trichotomy condition:** *For all  $a$  and  $b$ , exactly one of the following alternatives holds:*

$$a < b, a = b, a > b.$$

- **transitive condition:** *For all  $a$  and  $b$ ,*

$$a < b \wedge b < c \rightarrow a < c$$

- **isotonic condition for addition:** For all  $a, b$  and  $c$ ,

$$b < c \rightarrow a + b < a + c.$$

- **isotonic conditions for multiplication:** For all  $a, b$  and  $c$ ,

$$0 < a \wedge b < c \rightarrow a \cdot b < a \cdot c$$

and

$$0 > a \wedge b < c \rightarrow a \cdot b > a \cdot c.$$

**Definition:** Let  $A$  be an ordered ring. Then the **absolute value** of an element  $a$ , denoted  $|a|$ , is defined by  $|a| := a$  if  $0 < a$ ,  $:= 0$  if  $a = 0$  and  $:= -a$  if  $0 < -a$ .

**Proposition 10.** *The absolute value function satisfies the following conditions:*

- **triangle inequality:**  $|a + b| \leq |a| + |b|$ ;
- $|a \cdot b| = |a| \cdot |b|$ .

**Proposition 11.** *Let  $D$  be an ordered domain and let  $\text{Frac}(D)$  be the fraction field (that is, field of quotients). Then there is one and only one way to define an order on  $\text{Frac}(D)$  so that the standard injection  $D \rightarrow \text{Frac}(D)$  is an order morphism. In particular, this order is defined by  $0 < a/b$  (in  $\text{Frac}(D)$ ) if  $0 < a \cdot b$  (in  $D$ ).*

*Proof.* I shall regard  $D$  as a subset of  $\text{Frac}(D)$ .

The definition of the order in  $\text{Frac}(D)$  is motivated by the following equation:  $a/b = ab(1/b)^2$ . Recall that squares of nonzero elements are positive in ordered rings. Hence if  $0 < ab$  in  $D$  then  $a/b$  must be positive. These remarks determine uniqueness.

*Claim:* The indicated order is well-defined; in symbols,  $a/b = c/d \rightarrow (0 < ab \iff 0 < cd)$ .

Note that the following conditions are equivalent:  $a/b = c/d, ad = bc, ad(bd) = bc(bd), abd^2 = b^2cd$ .

*Claim:* The order on  $\text{Frac}(D)$  is isotonic for multiplication; in symbols,  $0 < a/b \wedge 0 < c/d \rightarrow 0 < (a/b)(c/d)$ .

Note  $(ac)(bd) = (ab)(cd)$ .

*Claim:* The order on  $\text{Frac}(D)$  is isotonic for addition; in symbols,  $0 < a/b \wedge 0 < c/d \rightarrow 0 < (a/b) + (c/d)$ .

Note  $bd(ad + bc) = abd^2 + b^2cd$ .

*Claim:* The order on  $\text{Frac}(D)$  satisfies the trichotomy condition.

Consider any  $a/b \neq 0$ . If  $0 < ab$  then  $0 < a/b$ . If  $0 < -ab$  then  $0 < -a/b$ .  $\square$

Recall that there is only one way to order the integers. It follows that there is only one way to order the rational numbers such that the injection of the integers into the rational numbers is an order morphism. Loosely speaking, there is only one way to order the rational numbers.

**Proposition 12.** *Let  $F$  be an ordered field. Then*

- For all  $a$ ,

$$0 < 1/a \iff 0 < a.$$

- For all  $a, b, c, d$ ,

$$a/b < c/d \iff abd^2 < b^2cd.$$

- For all  $a, b$ ,

$$0 < a < b \rightarrow 0 < 1/b < 1/a.$$

- For all  $a_1, a_2, \dots, a_n$ ,

$$0 < a_1^2 + a_2^2 + \dots + a_n^2 \quad \text{unless} \quad a_1 = a_2 = \dots = a_n = 0.$$

Recall the universal property of the fraction field: Let  $D$  be an integral domain let  $F$  be a field and let  $h : D \rightarrow F$  be a ring homomorphism. Then there is a unique ring homomorphism  $\hat{h} : \text{Frac}(D) \rightarrow F$  which extends  $h$ .

**Proposition 13.** *If  $D$  and  $F$  are ordered and  $h$  is an order morphism then so is  $\hat{h}$ .*

*Proof.* Recall  $\hat{h}(a/b) = h(a)/h(b)$ . Note  $0 < a/b \rightarrow 0 < ab$  and hence  $0 < h(ab) = h(a)h(b)$  and  $0 < h(a)/h(b)$ .  $\square$