A FIRST ORDER LANGUAGE FOR ORDERED RINGS

KENNETH R. DRIESSEL

I mainly follow Lyndon(1966) . Here is another reference for this material: Schoenfeld(1967) .

Date: October 8, 2006.

We consider mathematical structures of the following type:

$$\mathcal{A} := (A, +, 0, \cdot, 1, -1, \leq)$$

where A is a set, $+$ and $\cdot$ are binary operations on A, $0, 1$ and -1 are distinguished elements of A, and $\leq$ is a binary relation on A. (Note, for example, that rings and fields are structures of this type.) With such a structure we shall associate a first order language.

We shall discuss both the syntax and semantics of this language. We shall begin with syntax. In particular, we shall discuss the grammar of the language – that is, the rules used to determine the terms and (well-formed) formulas of the language. Then we shall discuss the semantics of the language – that is, the meaning of the terms and formulas. Our discussion will take place in an informal metalanguage; we use the usual language of mathematics, namely, informal set theory.

With a structure of the given type, we associate a **first-order language** L defined as follows: The **symbols** of the language L consist of mathematical symbols and logical symbols. The **mathematical symbols** are the **addition symbol** $+$, the **multiplication symbol** $\cdot$, the **constant symbols** $0, 1, -1$, and the **comparison symbol** $\leq$. The **logical symbols** are the **the grouping symbols** (and), the **disjunction symbol** $\vee$, the **conjunction symbol** $\wedge$, the **negation symbol** $\neg$, the **variables** $x_0, x_1, x_2, \ldots,$ the **existential quantifier** $\exists$, the **universal quantifier** $\forall$, and the **equality symbol** $=$. Finite strings of symbols are **expressions** of L.

The set Term of **terms** of L is the smallest set of expressions satisfying the conditions:

- The constant symbols are terms.
- The variables are terms.
- If α and β are terms then so are the **sum** $(\alpha + \beta)$ and the **product** $(\alpha \cdot \beta)$.

The set Form of **formulas** of L is the smallest set of expressions satisfying the conditions:

- If α and β are terms then the **comparison** $\alpha \leq \beta$ and the **equation** $\alpha = \beta$ are **atomic** formulas.
- If p and q are formulas then the **disjunction** $(p \vee q)$ and the **conjunction** $(p \wedge q)$ are formulas.
- If p is a formula then the **negation** $(\neg p)$ is a formula.
- If p is a formula and x is a variable then the expression $(\exists x)p$ is an **existentially quantified** formula and the expression $(\forall x)p$ is a **universally quantified** formula.

Proposition 1. Unique readability for terms. *Let γ be a term. Then exactly one of the following conditions holds:*

- *There is a unique constant such that γ is this constant.*
- *There is a unique variable such that γ is this variable.*
- *There are unique terms α and β such that γ is the sum expression $(\alpha + \beta)$.*
- *There are unique terms α and β such that γ is the product expression $(\alpha \cdot \beta)$.*

Proposition 2. Unique readability for formulas. *Let p be a formula. Then exactly one of the following conditions holds:*

- *The formula p is a comparison and there are unique terms α and β such that p is the expression $\alpha \leq \beta$.*
- *The formula p is a conjunction and there are unique formulas p_1 and p_2 such that p is the expression $(p_1 \vee p_2)$.*
- *The formula p is a disjunction and there are unique formulas p_1 and p_2 such that p is the expression $(p_1 \wedge p_2)$.*
- *The formula p is a negation and there exists a unique formula q such that q is the expression $(\neg q)$.*
- *The formula p is an existentially quantified formula and there exists a unique formula q and a unique variable x such that p is the expression $(\exists x)q$.*
- *The formula p is a universally quantified formula and there exists a unique formula q and a unique variable x such that p is the expression $(\forall x)q$.*

With every term and every formula we associate a set of **free variables**. In particular, we define the set $\mathrm{Free}(\alpha)$ where α is a term as follows:

- If α is a constant then $\mathrm{Free}(\alpha)$ is the empty set.
- If α is the variable x then $\mathrm{Free}(\alpha) := \{x\}$.
- If α and β are terms then $\mathrm{Free}(\alpha+\beta) := \mathrm{Free}(\alpha\cdot\beta) := \mathrm{Free}(\alpha) \cup \mathrm{Free}(\beta)$.

We define the set $\mathrm{Free}(p)$ where p is a formula as follows:

- $\mathrm{Free}(\alpha \leq \beta) := \mathrm{Free}(\alpha = \beta) := \mathrm{Free}(\alpha) \cup \mathrm{Free}(\beta)$.
- $\mathrm{Free}(p \vee q) := \mathrm{Free}(p \wedge q) := \mathrm{Free}(p) \cup \mathrm{Free}(q)$.
- $\mathrm{Free}(\neg p) := \mathrm{Free}(p)$.
- $\mathrm{Free}((\exists x)p) := \mathrm{Free}((\forall x)p) := \{y \in \mathrm{Free}(p) : y \neq x\}$.

A term with no free variables is a **variable-free term**. For example, the expression $(((1+1)+1)+1)$ is a variable-free term. (The expression 4 is a common abbreviation for this term.) Formulas with no free variable are called **sentences** or **closed formulas**.

We now turn to a discussion of semantics. Let L denote the language that we just defined. We define an **interpretation** of L in a structure

$$\mathcal{A} := (A, +, 0, \cdot, 1, -1, \leq)$$

as follows: Let $\phi : \mathrm{Var} \to A$ be a function where $\mathrm{Var} := \{x_0, x_1, \dots\}$ is the set of variables of L. We extend ϕ to the set Term of terms by induction: $\phi(0) := 0$, $\phi(1) := 1$, $\phi(-1) := -1$; if α and β are terms then $\phi(\alpha + \beta) := \phi(\alpha) + \phi(\beta)$ and $\phi(\alpha \cdot \beta) := \phi(\alpha) \cdot \phi(\beta)$. Note that terms provide names of elements of A via the interpretation map ϕ. We also extend ϕ to the set Form of formulas by induction: If α and β are terms then $\phi(\alpha = \beta) := T$ if $\phi(\alpha) = \phi(\beta)$ and $:= F$ otherwise and $\phi(\alpha \leq \beta) := T$ if $\phi(\alpha) \leq \phi(\beta)$ and $:= F$ otherwise; if p and q are formulas then $\phi(p \vee q) := \phi(p) \vee \phi(q)$, $\phi(p \wedge q) := \phi(p) \wedge \phi(q)$, and $\phi(\neg p) := \neg\phi(p)$. In other words, $\phi(p \vee q) := T$ iff $\phi(p) = T$ or $\phi(q) = T$, etc. Here we use "T" and "F" as abbreviations of "true" and "false" respectively. Note that we allow a symbol to have different meanings depending on context. For example, the first $\vee$ in $\phi(p \vee q) := \phi(p) \vee \phi(q)$ is a formal symbol and the second one represents an operation on truth values. We still need to consider quantified formulas. For a variable x let $I(\phi, x)$ be the set of mappings $\psi : \mathrm{Var} \to A$ which agree with ϕ except possibly at x. Then we define the extension of ϕ to quantified formulas as follows: $\phi((\forall x)p) := T$ if for all $\psi \in I(\phi, x)$, $\psi(p) = T$ and $:= F$ otherwise; $\phi((\exists x)p) := T$ if for some $\psi \in I(\phi, x)$, $\psi(p) = T$ and $:= F$ otherwise.