ELIMINATION OF QUANTIFIERS

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I mainly follow Tarski(1951,1967) and the book by Basu, Pollack and Roy(2003) .

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As usual we consider structures having the following type:

$$\mathcal{A} = (A, +, 0, \cdot, 1, -1, \leq)$$

where A is a set, $+$ and $\cdot$ are binary operations on A, $0, 1$ and -1 are elements of A and $\leq$ is a binary relation on A. In other words, $\mathcal{A}$ has the same type as an ordered ring. We also consider the related first order language which we denote by L.

Definition: Let $\mathcal{A}$ be a structure with the given type and let p be a formula in L. Then we say that p is **satisfied in** $\mathcal{A}$ or that $\mathcal{A}$ **models** p, which we write $\mathcal{A} \models p$, if $\phi(p) = T$ for every interpretation ϕ in $\mathcal{A}$.

The binary relation $\models$ determines a Galois correspondence between classes of structures and sets of formulas: With a class $\mathbb{A}$ of structures, we associate the following set of formulas:

$$\mathbb{A}^* := \{p : (\forall \mathcal{A} \in \mathbb{A}) \quad \mathcal{A} \models p\}.$$

With a set P of formulas, we associate the following class of structures:

$$P^* := \{\mathcal{A} : (\forall p \in P) \quad \mathcal{A} \models p\}.$$

We say that $\mathbb{A}^*$ is the **(closed) theory** determined by $\mathbb{A}$. And we say that P^* is the **axiomatic class** of structures determined by P.

We shall mainly be concerned with the class of all stuctures with the given type, the class of all ordered rings and the class of all real closed fields.

We shall write $\alpha(Y_1, \ldots, Y_n)$ to mean that the free variables in the term α are among those in the set $\{Y_1, \ldots, Y_n\}$. We shall also write $p(Y_1, \ldots, Y_n)$ to mean that the free variables in the formula p are among those in the set $\{Y_1, \ldots, Y_n\}$.

Definition: Let T be a theory in the first order language L. Let $\alpha(Y_1, \ldots, Y_n)$ and $\beta(Y_1, \ldots, Y_n)$ be terms in L. Then α and β are **equivalent in** T if the formula

$$(\forall Y_1) \ldots (\forall Y_n) \quad (\alpha(Y_1, \ldots, Y_n) = \beta(Y_1, \ldots, Y_n))$$

is in T. Let $p(Y_1, \ldots, Y_n)$ and $q(Y_1, \ldots, Y_n)$ be formulas in L. Then p and q are **equivalent in** T if the formula

$$(\forall Y_1) \ldots (\forall Y_n) \quad (p(Y_1, \ldots, Y_n) \leftrightarrow q(Y_1, \ldots, Y_n))$$

is in T.

We use $p \rightarrow q$ (**logical implication**) as an abbreviation for $\neg p \vee q$ and $p \leftrightarrow q$ (**logical equivalence**) as an abbreviation for $(p \rightarrow q) \wedge (q \rightarrow p)$.

The following two propositions are true for the theory of all structures of the given type.

Proposition 1. *The relation of equivalence between terms has the following properties:*

- *The relation is an equivalence relation (symmetric, reflexive and transitive).*

- *If* α_1 *and* α_2 *are equivalent terms, and if the term* β_2 *arises from the term* β_1 *by replacing* α_1 *by* α_2 *at one or more places, then* β_1 *is equivalent to* β_2*.*
- *If* α_1 *and* α_2 *are equivalent terms, and if the formula* p_2 *arises from the formula* p_1 *by replacing* α_1 *by* α_2 *at one or more places, then* p_1 *is equivalent to* p_2*.*

Proposition 2. *The relation of equivalence between formulas has the following properties:*

- *The relation is an equivalence relation.*
- *Let* p_1 *and* p_2 *be equivalent formulas, and suppose that the formula* q_2 *arises from the formula* q_1 *by replacing* p_1 *by* p_2 *at one or more places. Then* q_1 *is equivalent to* q_2*.*

We shall use the following two results about equivalence later.

Proposition 3. Disjunctive normal form. *Let* T *be any theory in* L*. Let* $p(Y_1, \ldots, Y_n)$ *be a formula which involves no negation signs or quantifiers. Then this formula is equivalent in* T *to a formula* $q(Y_1, \ldots, Y_n)$ *with the same free variables which is a disjunction of conjunctions of atomic formulas.*

Proof. (Sketch) Consider the following production rule:

- $p \wedge (q \vee r) \mapsto (p \wedge q) \vee (p \wedge r)$.

(This rule is associated with the distributive law of conjunction over disjunction.) □

Proposition 4. Distributive law for existential quantification over disjunction. *Let* T *be any theory in* L*. Let* p *and* q *be formulas in* L *and let* X *be a variable. Then the following two formulas are equivalent in* L*:*

- $(\exists X)(p \vee q)$
- $(\exists X)p \vee (\exists X)q$.

Definition: A theory T in a first order language L **admits elimination of quantifiers** if every formula in L is equivalent in T to a quantifier free formula.

The following result shows that if we are trying to show that a theory admits elimination of quantifiers then we can concentrate on formulas of a particular type called "simply existential formulas".

Definition: A formula in a first order language is **simply existential** if it has the form $(\exists X)\ p$ where p is a quantifier free formula.

Proposition 5. Reduction theorem for elimination of quantifiers. *Let* T *be any theory in the first order language* L*. If every simply existential formula in* L *is equivalent in* T *to a quantifier free formula then* T *admits elimination of quantifiers.*

Proof. Let p be a formula in L. We need to find a quantifier free formula q that is equivalent to p. If p is quantifier free then we can take q to be p itself. Otherwise consider the subformulas of p. Since p is built up using conjunction $\wedge$, disjunction $\vee$, negation $\neg$ and existential quantification from atomic formulas (which are quantifier free) there is a subformula p' of p which is simply existential. (Here we regard the universal quantifier $\forall X$ as an abbreviation for $\neg\,(\exists X)\,\neg$.) By hypothesis the subformula p' is equivalent in T to a formula q' which is quantifier free. Substitute q' for p' in p to get an equivalent formula that has one less existential quantifier. Continue in this way until the resulting formula is quantifier free. $\square$

The last result shows that we should concentrate our attention on simply existential formulas, namely, ones having the form $(\exists X)p$ where p is a quantifier free formula. The following results show that in the theory of order rings we need only consider quantifier free formulas having the following form:

$$f = 0 \wedge g_1 > 0 \wedge g_2 > 0 \cdots \wedge g_k > 0$$

where the f and g_i are multivariate polynomials.

Proposition 6. *Let T be the theory of ordered rings. Then every quantifier free formula $p(Y_1, \ldots, Y_n)$ is equivalent in T to a formula $q(Y_1, \ldots, Y_n)$ with the same free variables built up by means of conjunction and disjunction from atomic formulas having the following form: $\alpha = 0$, $\beta > 0$.*

Proof. Consider the following production (or rewrite) rules where α and β are terms and r and s are formulas:

- $\alpha = \beta \mapsto \alpha - \beta = 0$
- $\alpha > \beta \mapsto \alpha - \beta > 0$
- $\neg(\alpha = \beta) \mapsto \alpha > \beta \vee \alpha < \beta$
- $\neg(\alpha > \beta) \mapsto \alpha = \beta \vee \alpha < \beta$
- $\neg\neg r \mapsto r$
- $\neg(r \vee s) \mapsto \neg r \wedge \neg s$
- $\neg(r \wedge s) \mapsto \neg r \vee \neg s$.

(The last two of these rules are associated with De Morgan's laws.) Note that the application of any of these rules to a subformula of p transforms p into an equivalent formula. Also note that the number of free variables does not change. Also note that the number of negations remains the same or decreases. It is clear that the sucessive applications of these rules leads to a formula described in the conclusion of the proposition. $\square$

Proposition 7. *Let T be the theory of ordered rings. Let $\alpha(Y_1, \ldots, Y_n)$ be a term. Then $\alpha(Y_1, \ldots, Y_n)$ is equivalent in T to a multivariate polynomial $f(Y_1, \ldots, Y_n)$ with the same free variables.*

Proof. (Sketch) Use the following production rule:

- $\alpha \cdot (\beta + \gamma) \mapsto \alpha \cdot \beta + \alpha \cdot \gamma$.

We omit the details. $\square$

Proposition 8. *Let T be the theory of ordered rings. Let $\alpha_1, \ldots, \alpha_k$ be terms. Let $\beta := \alpha_1^2 + \cdots + \alpha_k^2$. Then the following two formulas are equivalent in T:*

- $\alpha_1 = 0 \wedge \cdots \wedge \alpha_k = 0$
- $\beta = 0$.