

DEFINABLE SETS

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We mainly follow Basu, Pollack and Roy(2003) .

Let $\mathcal{A} := (A, +, 0, \cdot, 1, -1, \leq)$ be an ordered ring and let L be the corresponding first order language. Let Variables denote the set of variables in L . Recall that we can extend a map $\phi : \text{Variables} \rightarrow A$ to an interpretation of L in \mathcal{A} . In particular, the interpretation sends terms of L to elements of A and sends formulas of L to the truth values T and F . We shall also use ϕ to denote the interpretation determined by ϕ .

The following proposition says that the value of an interpretation of a term or formula only depends on the value of the interpretation on the free variables occurring in the term or formula.

Proposition 1. *Let ϕ and ψ be interpretations of L in an ordered ring \mathcal{A} which agree on the set of variables $X := \{X_1, \dots, X_n\}$. If α a term with all its free variables in the set X then $\phi(\alpha) = \psi(\alpha)$. If p is a formula with all its free variables in the set X then $\phi(p) = \psi(p)$.*

Definition: Let $\alpha(X_1, \dots, X_n)$ be a term with all its free variables in the set $X := \{X_1, \dots, X_n\}$ and let $p(X_1, \dots, X_n)$ be a formula with all its free variables in the set X . Let a_1, \dots, a_n be elements of the ordered ring A . We use $\alpha(a_1, \dots, a_n)$ to denote the value of α for any interpretation that maps X_i to a_i for $i = 1, \dots, n$. We also use $p(a_1, \dots, a_n)$ to denote the value of p for any such interpretation. The A -**Realization** of p , denoted $\text{Reali}(p, A^n)$, is defined by

$$\text{Reali}(p, A^n) := \{y \in A^n : p(y)\}.$$

Note we have the following properties (which can be used to provide an inductive definition of the realization operator):

- $\text{Reali}(\alpha = 0, A^n) = \{y \in A^n : \alpha(y) = 0\}$
- $\text{Reali}(\alpha > 0, A^n) = \{y \in A^n : \alpha(y) > 0\}$
- $\text{Reali}(p \wedge q, A^n) = \text{Reali}(p, A^n) \cap \text{Reali}(q, A^n)$
- $\text{Reali}(p \vee q, A^n) = \text{Reali}(p, A^n) \cup \text{Reali}(q, A^n)$
- $\text{Reali}((\exists X)p, A^n) = \{y \in A^n : (\exists x \in A) (x, y) \in \text{Reali}(p, A^{n+1})\}$
- $\text{Reali}((\forall X)p, A^n) = \{y \in A^n : (\forall x \in A) (x, y) \in \text{Reali}(p, A^{n+1})\}.$

Two formulas $p(X_1, \dots, X_n)$ and $q(X_1, \dots, X_n)$ are A -**equivalent** if they have the same realization: $\text{Reali}(p, A^n) = \text{Reali}(q, A^n)$. A subset S of A^n is **definable** if there is a formula $p(X_1, \dots, X_n)$ such that $S = \text{Reali}(p, A^n)$. The subset S is **semi-algebraic** if there is a quantifier free formula p such that $S = \text{Reali}(p, A^n)$.

Here are some examples: $\text{Reali}(0 < 1, A^n) = A^n$ and $\text{Reali}(0 > 1, A^n) = \emptyset$. More generally, if p is any formula which is true in A , then $\text{Reali}(p, A^n) = A^n$. If $p(X, Y)$ is a formula with two variables then we can draw a picture representing $\text{Reali}(p, A^2)$; in particular, we can represent A^2 as a plane and we can then represent $\text{Reali}(p, A^2)$ as the region in the plane consisting of the pairs (x, y) at which p is true. Note that we can then also represent $\text{Reali}((\exists Y)p(X, Y), A)$ as the projection on the x -axis of the region representing $\text{Reali}(p, A^2)$.

Let T be (for example) the theory of ordered rings. Note that if p and q are formulas which are equivalent in T and A is an ordered ring, then these two formulas are A -equivalent.

Note that every semi-algebraic set is definable. In the theory of real closed fields every formula is equivalent to a quantifier free formula. (That is Tarski's quantifier elimination theorem for real closed fields.) Hence for real closed fields every definable set is semi-algebraic.

Remark: The definable subsets associated with a ring provide an example of a 'cylindric algebra'. For more on such algebras see Henkin, Monk and Tarski(1971) .