# REAL ALGEBRAIC GEOMETRY

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## Decomposition of Algebras Using Idempotents

*References:* In this section I mainly follow the books Basu, Pollack and Roy(2003) and Cox, Little and O'Shea(1991,1998).

*Notation:* Throughout this section we use the following notation: $K$ denotes a field and $A$ denotes a commutative algebra over $K$ with identity element 1.

*Example:* Let $M$ be a square matrix with elements in $K$. Then the set of matrices which are polynomials in $M$ forms a commutative algebra with identity; in other words, the set $\{p(M) : p \in K[X]\}$ is such an algebra.

We can often decompose algebras into simpler subalgebras. The following general result provides a method for doing such decompositions. (The following directive is a standard guideline in the theory of algebras: "Look for idempotents.")

**Proposition 1. Decomposition of an algebra using idempotents.** *Let $e_1, \ldots, e_n$ be elements of algebra $A$ which satisfy the following conditions:*

- $e_1 + e_2 + \cdots + e_n = 1$ *and*
- *if* $i \neq j$ *then* $e_i e_j = 0$.

*Then*

- $e_i^2 = e_i$,
- $e_i A$ *is a subring of* $A$ *with identity, namely,* $e_i$, *and*
- $A$ *is the direct sum of the subrings* $e_i A$*:*

$$A = e_1 A \oplus e_2 A \oplus \ldots e_n A.$$

*Definition:* A set of elements $e_1, \ldots, e_n$ which satisfies the hypothesis of the proposition is called an **orthogonal set of idempotents for** $A$.

*Example:* Here is an example. Let Diag(4) denote the set of diagonal $4 \times 4$ matrices. Note that this set may be regarded as a commutative algebra with identity. Note that the diagonal matrices $e_1 := \mathrm{diag}(1, 1, 1, 0)$ and $e_2 := \mathrm{diag}(0, 0, 0, 1)$ provide a set of orthogonal idempotents for this algebra.

*Proof.* In order to simplify the notation we do the case $n = 2$. It should be clear that this proof generalizes to arbitrary $n$.

*Claim:* $e_1^2 = e_1$ and $e_2^2 = e_2$.

*Date*: February 4, 2007.

We simply calculate as follows:

$$e_1^2 = e_1e_1 + e_1e_2 = e_1(e_1 + e_2) = e_11 = e_1.$$

This shows that $e_1$ is an idempotent. To see that $e_2$ is an idempotent simply interchange 1 and 2 in the calculation.

*Claim:* The sets $e_1A$ and $e_2A$ are subrings of $A$.

We need to see that $e_1A$ is closed under scalar multiplication, addition and multipication. We have, for any scalar $\alpha$, $\alpha(e_1a) = e_1(\alpha a)$. We also have $e_1a + e_1b = e_1(a + b)$ and $(e_1a)(e_1b) = e_1(ae_1b)$. The proof that $e_2A$ is a subring is similar.

*Claim:* $A = e_1A + e_2A$.

Note $a = (e_1 + e_2)a = e_1a + e_2a$.

*Claim:* $e_1A \cap e_2A = 0$.

Let $c$ be in the intersection. Then there exist elements $a$ and $b$ in $A$ such that $c = e_1a = e_2b$. Then $e_1a = e_1^2a = e_1(e_2b) = (e_1e_2)b = 0$. □

**Definition.** Note that the map $A \times A \to A : (f, g) \mapsto fg$ is a bilinear map on $A$. For $f$ in $A$ we define the map $L(f)$ by

$$L(f) := A \to A : g \mapsto fg.$$

This map is called the **multiplication map associated with** $f$. Note that $L(f)$ is a linear map on $A$. In other words, we have a map

$$L : A \to \mathrm{Lin}(A \to A) : f \mapsto L(f)$$

where $\mathrm{Lin}(A \to A)$ is the algebra of linear maps on $A$.

The following proposition says that the map $L$ is a representation of $A$. (One of the standard guidelines in the study of algebras is: "Look at representations".) Note that the image of $A$ under $L$ is a commuting algebra of linear transformations.

*Definition.* I call the map $L$ the **standard representation of** $A$.

**Proposition 2. Standard representation of an algebra.** *The map $L$ is an injective homomorphism from the algebra $A$ into the algebra $Lin(A \to A)$.*

*Proof.* It is easy to see that $L$ is a linear map. We also have $L(f_1f_2) = L(f_1) \circ L(f_2)$ since $L(f_1f_2)(g) = f_1f_2g = (L(f_1) \circ L(f_2))(g)$. To finish the proof we only need to show that $L$ is injective. Suppose $L(f) = 0$. Then for every $g$ in $A$ we have $fg = 0$. In particular, $f = f \cdot 1 = 0$. □

We can use idempotents to decompose the standard representation $L$.

**Proposition 3. Decomposition of the standard representation.** *Let $e_1, e_2, \ldots, e_n$ be a set of orthogonal idempotents for $A$ and let $f$ be an element of $A$. Then the subalgebras $e_iA$ are invariant subspaces of the linear map $L(f)$ and*

$$L(f) = L(e_1f) \oplus L(e_2f) \oplus \cdots \oplus L(e_nf).$$

*Proof.* Consider any element $e_ia$ of $e_iA$. We have $L(f)(e_ia) = e_i(fa)$. We also have $L(f)g = fg = e_1fg + \cdots + e_nfg = (L(e_1f) + \cdots + L(e_nf))g$. □

If we choose bases $B_i$ for the subalgebras $e_iA$ and put these bases together to get a basis $B$ for $A$ then the matrix of $L(f)$ with respect to the basis $B$ is a block matrix. The blocks are matrices of the restricted linear maps $L(e_if)$ with respect to the $B_i$.

**Proposition 4. The standard bilinear form on an algebra.** *Assume that $A$ is finite dimensional. Let $q$ be an element of $A$. Then the function*

$$A \times A \to K : (f, g) \mapsto \mathit{Trace}(L(qfg))$$

*is a symmetric bilinear form and the function*

$$A \to K : f \mapsto \mathit{Trace}(L(qf^2))$$

*is a quadratic form.*

*Definition:* I call the first of these functions the **standard bilinear map on $A$ determined by** $q$ and the second one the **standard quadratic form on $A$ determined by** $q$.

*Proof.* Let $B(f, g) := \mathrm{Trace}(L(qfg))$. We need to see that $B(f_1 + f_2, g) = B(f_1, g) + B(f_2, g)$ and $B(\alpha f, g) = \alpha B(f, g)$. The proofs of these identities are straightforward; in particular, they follows from the linearity of the Trace operator and the linearity of the representation $L$. We also need to see that $B(f, g) = B(g, f)$. This follows easily since $A$ is commutative. □

We can use idempotents to decompose the standard bilinear form.

**Proposition 5. Decomposition of the standard bilinear forms.** *Let $A$ be a finite dimensional algebra. Let $e_1, \ldots, e_n$ be a set of orthogonal idempotents for $A$. Let $q$ be an element of $A$. And let $B_q$ denote the standard bilinear form on $A$ determined by $q$. Then $i \neq j$ implies $B_q(e_iA, e_jA) = 0$, that is, for every pair of elements $a \in e_iA$ and $b \in e_jA$, $B_q(a, b) = 0$.*

*Proof.* We have $\mathrm{Trace}(L(e_ixe_jy)) = \mathrm{Trace}(L(0)) = 0$. □

Suppose we choose bases $B_i$ of the subalgebras $e_iA$ and put them together to get a basis $B$ of $A$. Then the matrix $M$ of a standard bilinear form on $A$ with respect to $B$ is a block matrix. The $i$th block is a matrix of the restricted bilinear form $e_iA \times e_iA \to K : (e_if, e_ig) \mapsto \mathrm{Trace}(L(qe_ife_ig))$ with respect to $B_i$.