REAL ALGEBRAIC GEOMETRY

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Cones and Homomorphisms

References: In this section I mainly follow Prestel and Delzell(2001) .

Notation: Throughout this section, we use the following notation: A denotes a commutative ring with identity 1, A^2 denotes the set of squares of elements of A, $\sum A^2$ denotes the set of sums of squares of elements of A.

Proposition 1. Inverse images of cones. *Let A and B be commutative rings with identity. Let $\alpha : A \to B$ be a ring homomorphism. Let U be a subset of B and let $T := \alpha^{-1}(U) := \{a \in A : \alpha(a) \in U\}$.*

- *If B^2 is a subset of U then A^2 is a subset of T.*
- *If U is closed under addition then so is T.*
- *If U is closed under multiplication then so is T.*
- *If -1 is not an element of U then -1 not an element of T.*
- *If U is a pre-positive cone of B then T is a pre-positive cone of A.*
- *If U is total in B (that is, $B = U \cup -U$) then T is total in A.*

Furthermore, if Q is a positive cone in B then $P := \alpha^{-1}(Q)$ is a positive cone in A.

Proof. Assume B^2 is a subset of U. Consider any a in A. We have $\alpha(a^2) = \alpha(a)^2 \in B^2 \subseteq U$. Hence $a^2 \in T$.

Assume U is closed under addition. Consider any two elements a and b in T. We have $\alpha(a+b) = \alpha(a) + \alpha(b) \in U$. Hence $a+b$ is an element of T.

Assume U is closed under multiplication. Consider any two elements a of b in T. We have $\alpha(ab) = \alpha(a)\alpha(b) \in U$. Hence ab is ann element of T.

Suppose -1 is an element of T. Then $-1 = \alpha(-1)$ is an element of U.

Assume that U is a pre-positive cone in B. Then T is a pre-positive cone in A by the previous parts of this proof.

Assume U is total in B. Consider any a in A. Note $\alpha(a)$ is an element of $U \cup -U$. In other words, either $\alpha(a)$ or $\alpha(-a) = -\alpha(a)$ is an element of U. Hence a or $-a$ is an element of T.

Now assume that Q is a positive cone in B. We need to see that the support $\text{Supp}(P) := P \cap -P$ is a prime ideal in A. We already know that P is a total pre-positive cone in A. Hence the support of P is an ideal. (We proved this earlier.) We need to see that it is prime. Consider any pair of elements a and b of A with $ab \in P \cap -P$. Note $\alpha(a)\alpha(b) = \alpha(ab) \in Q$ and

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$-\alpha(a)\alpha(b) = -\alpha(ab) \in Q$. hence $\alpha(a)\alpha(b) \in \mathrm{Supp}(Q)$. Since $\mathrm{Supp}(Q)$ is a prime ideal, we conclude that either $\alpha(a) \in \mathrm{Supp}(Q)$ or $\alpha(b) \in \mathrm{Supp}(Q)$. Hence either $a \in \mathrm{Supp}(P)$ or $b \in \mathrm{Supp}(P)$.

Remark: There is an alternative to the last part of the proof. We could note that $\alpha^{-1}(Q \cap -Q) = P \cap -P$ and also note that the inverse image of a prime ideal is a prime ideal. □

Proposition 2. Image of a pre-positive cone. *Let A and B be commutative rings and let $\alpha : A \to B$ be a surjective ring homomorphism. Let T be a subset of A and let U be its image in B, that is, $U := \alpha(T) := \{\alpha(t) : t \in T\}$.*

- *If A^2 is a subset of T then B^2 is a subset of U.*
- *If T is closed under addition then so is U.*
- *If T is closed under multiplication then so is U.*
- *If T is total in A then U is total in B.*

Assume further that the kernel of α is a subset of T: Kernel$(\alpha) \subseteq T$. Then if T is a pre-positive cone, so is U.

Proof. Assume A^2 is a subset of T. Consider any b in B. Since α is surjective there is an a in A with $\alpha(a) = b$. Then $b^2 = \alpha(a)^2 = \alpha(a^2)$ is in U.

Assume T is closed under addition. Consider any pair of elements u and v of U. There exist elements p and q in T such that $\alpha(p) = u$ and $\alpha(q) = v$. Then $u + v = \alpha(p + q)$ is an element of U.

Assume T is closed under multiplication. Consider any pair of elements u and v of U. There exist elements p and q in T such that $\alpha(p) = u$ and $\alpha(q) = v$. Then $uv = \alpha(pq)$ is an element of U.

Assume that the kernel of α is a subset of T.

We need to see that if -1 is not an element of T then -1 is not an element of U. Suppose -1 is an element of U. Then there is an element p of T such that $\alpha(p) = -1 = \alpha(-1)$. Hence $q := p + 1$ is in the kernel of α. Note $-q$ is also an element of this kernel (because it is an ideal). Hence $-q$ is an element of T and $-1 = p - q$ is an element of T. □

Example: Note that thte hypothesis concerning the kernel of α in the last result is necessary. For example, consider the canonical homomorphism $\mathbb{R}[X] \to \mathbb{R}[X]/\mathrm{Ideal}(X^2 + 1)$. The image is isomorphic to the field $\mathbb{C}$ of complex numbers. This field has no pre-positive cones. But the ring $\mathbb{R}[X]$ does have prepositive cones.

Proposition 3. Image of a positive cone. *Let P be positive cone in A. Let $\alpha : A \to \overline{A} := A/Supp(P)$ be defined by $a \mapsto a + Supp(P)$. Then $\overline{P} := \alpha(P)$ is a positive cone in $A/Supp(P)$ with support equal to 0.*

Proof. Note that $\overline{P}$ is a pre-positive cone by the last proposition.

Claim: $\overline{P} \cap -\overline{P} = 0$

Note $\overline{P} \cap -\overline{P} = \alpha(P \cap -P) = 0$.

Claim: The zero ideal in $\overline{A}$ is prime.

Note that $\overline{A}$ is an integral domain since $\mathrm{Supp}(P)$ is a prime ideal. □