CLOSURE OPERATORS AND GALOIS CONNECTIONS

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I mainly follow Cohn(1965) .

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Definition: Let A be a set. Let Boole(A) denote the collection of all subsets of A; in symbols,

$$\text{Boole}(A) := \{X : X \subseteq A\}.$$

Let $\mathcal{C}$ be a subset of Boole(A) (that is, a collection of subsets of A). Then $\mathcal{C}$ is a **closure system** on A if $\mathcal{C}$ is closed under intersection: in symbols,

$$\mathcal{D} \subseteq \mathcal{C} \Rightarrow \cap\mathcal{D} \in \mathcal{C}$$

where

$$\cap\mathcal{D} := \{x \in A : \forall D \in \mathcal{D}(x \in D)\}.$$

Example: Let $(R, +, 0, \cdot, 1, -1)$ be a ring. Then the collection of all subrings of R forms a closure system on R. The collection of all ideals in R is also a closure system on R.

Definition: A mapping $J : \text{Boole}(A) \to \text{Boole}(A)$ is a **closure operator** on A if it satisfies the following conditions for all subsets X and Y of A:

- $X \subseteq Y \Rightarrow J(X) \subseteq J(Y)$,
- $X \subseteq J(X)$, and
- $JJ(X) = J(X)$.

A closure operator is **topological** if it satiesfies the following additional codition:

$$J(X \cup Y) = J(X) \cup J(Y).$$

Example: Let T be a topological space. Then we have the map $J : \text{Boole}(T) \to \text{Boole}(T)$ which sends a subset X of T to its closure. This map is a topological closure operator.

Example: Let $(R, +, 0, \cdot, 1, -1)$ be a ring. Let X be a subset of R. Then the smallest subring, denoted Ring(X), of R containing X is given by

$$\text{Ring}(X) := \cap\{R' : X \subseteq R' \text{ and } R' \text{ is a subring of } R\}.$$

This ring is called the **ring generated by** X. The map of Boole(R) into itself defined by $X \mapsto \text{Ring}(X)$ is a closure operator. The smallest ideal, denoted Ideal(X), of R containing X is given by

$$\text{Ideal}(X) := \cap\{I : X \subseteq I \text{ and } I \text{ is an ideal of } R\}.$$

This ideal is called the **ideal generated by** X. The map of Boole(R) into itself defined by $X \mapsto \text{Ideal}(X)$ is a closure operator.

Proposition 1. *Let A be a set.*

- *Let $\mathcal{C} \subseteq Boole(A)$ be a closure system on A. Define*

  $$J := Boole(A) \to Boole(A) : X \mapsto \cap\{Y \in \mathcal{C} : X \subseteq Y\}.$$

  Then J is a closure operator.

- *Let $J : \mathit{Boole}(A) \to \mathit{Boole}(A)$ be a closure operator. Define*
$$\mathcal{C} := \{X \subseteq A : J(X) = X\}.$$
Then $\mathcal{C}$ is a closure system on A.

Proof. Exercise. □

Remark: The last proposition shows that with every closure system $\mathcal{C}$ on A we can associate a closure operator $J(\mathcal{C})$ on A. It also shows that with every closure operator J on A we can associate a closure system $\mathcal{C}(J)$ on A. In this way we get a bijective correspondence between closure systems on A and closure operators on A.

Definition: Let $J : \mathrm{Boole}(A) \to \mathrm{Boole}(A)$ be a closure operator on A. Then J is **algebraic** if it satisfies the following condition:
$$\forall X \subseteq A, a \in A \ (a \in J(X) \Rightarrow \exists \text{ finite } Y \subset X, a \in J(Y)).$$
A closure system is **algebraic** if the corresponding closure operator is algebraic.

Example: Consider the real line $\mathbb{R}$ its usual topology. The associated closure operator is not algebraic. In particular, consider the open interval $(0,1)$. Its closure is the closed interval $[0,1]$. Note that 0 is in the closed interval but is not in the closure of any finite subset of the open interval.

Example: Recall that above we considered the closure system formed by the subrings of a ring. In the following proposition we show that this closure system is algebraic. (This example provides some motivation for the use of the word "algebraic".)

Proposition 2. *Let $(R, +, 0, \cdot, 1, -1)$ be a ring and let X be a subset of R. Define the following sequence X_k of subsets of R by induction:*

- $X_0 := X \cup \{0, 1, -1\}$*;*
- $X_{k+1} := X_k \cup \{x \in R : (\exists y, z \in X_k)(x = y + z \vee x = y \cdot z)\}$.

Then $\mathit{Ring}(X) = \cup\{X_k : k = 0, 1, \dots\}$. Consequently, the map
$$\mathit{Boole}(R) \to \mathit{Boole}(R) : X \mapsto \mathit{Ring}(X)$$
is an algebraic closure operator.

Proof. Let $U := \cup\{X_k : k = 0, 1, \dots\}$.

Claim: $U \subseteq \mathrm{Ring}(X)$.

An easy induction proof shows that $X_k \subseteq \mathrm{Ring}(X)$ for all k.

Claim: $U \supseteq \mathrm{Ring}(X)$.

Note that U is a subring of R which contains X.

Claim: If $X \subseteq Y$ then, for all k, $X_k \subseteq Y_k$.

Use induction.

Claim: $\forall k \forall a \in X_k \exists$ finite $X' \subseteq X (a \in X'_k)$.

We prove this assertion by induction on k.

$k = 0$: In this case we have $a \in X \cup \{0, 1, -1\}$. We can take $X' := \{a\}$. Then $a \in X'_0$.

$k+1$: Cosider an $a \in X_{k+1}$. If $a \in X_k$ then we can use the induction hypothesis. Suppose $a \notin X_k$. Then there exist b and c in X_k such that $a = b + c$ or $a = b \cdot c$. By the induction hypothesis there are finite subsets Y and Z of X such that $b \in Y_k$ and $c \in Z_k$. We take $X' := Y \cup Z$. $\square$

Remark: This proposition clearly generalizes to other algebraic structures.

We now define a "Galois connection". Such a connection provides important examples of closure operators.

Definition: Let A and B be sets and let $K : \mathrm{Boole}(A) \to \mathrm{Boole}(B)$ and $L : \mathrm{Boole}(B) \to \mathrm{Boole}(A)$ be maps. Then this pair of maps is a **Galois connection** (or **Galois correspondence**) between A and B if the following conditions are satisfied:

- $X_1 \subseteq X_2 \subseteq A \Rightarrow K(X_1) \supseteq K(X_2)$;
- $Y_1 \subseteq Y_2 \subseteq B \Rightarrow L(Y_1) \supseteq L(Y_2)$;
- $X \subseteq A \Rightarrow X \subseteq (L \circ K)(X)$;
- $Y \subseteq B \Rightarrow Y \subseteq (K \circ L)(Y)$.

Most Galois connections arise in the following way. Let A and B be sets. Let Φ be a relation between A and B; in symbols, $\Phi \subseteq A \times B$. For any subset X of A let

$$K(X) := \{y \in B : \forall x \in X((x, y) \in \Phi\}.$$

For any subset Y of B let

$$L(Y) := \{x \in A : \forall y \in Y((x, y) \in \Phi\}.$$

This pair of mappings provides a Galois correspondence between A and B.

Example: Let F be a field and let G be the group of all automorphisms of F. Consider the relation

$$\Phi := \{(x, \alpha) \in F \times G : \alpha(x) = x\}.$$

We obtain a Galois connection between subfields of F and subgroups of G. (In Galois theory, we study this particular Galois connection. The phrase "Galois connection" comes from this setting.)

Example: Let k be a field. Let $A := k[X_1, \ldots, X_n]$ be the ring of polynomials with coefficients in K and n variables. Let $B := k^n$. Let

$$\Phi := \{(f, x) \in A \times B : f(x) = 0\}.$$

For $F \subseteq A$ we have $K(F) = \{x \in B : (\forall f \in F) f(x) = 0\}$. In other words, $K(F)$ is the variety determined by the set of polynomial F. For $S \subseteq B$ we have $L(S) := \{f \in A : (\forall x \in S) f(x)\}$. In other words, $L(S)$ is the ideal of polynomials which anihilates the points in S. The next proposition shows that the map $K \circ L$ is a closure operator. Since the union of two varieties is a variety, we see that this closure operator is topological. It determines the Zariski topology on k^n.

Proposition 3. *Let A and B be sets and let K : Boole$(A) \to$ Boole(B) and L : Boole$(B) \to$ Boole(A) be maps which form a Galois connection. Then*

- $(\forall X \subseteq A)\ (K \circ L \circ K)(X) = K(X)$.
- $(\forall Y \subseteq B)\ (L \circ K \circ L)(Y) = L(Y)$.
- *The map* $L \circ K$ *is a closure operator on* A.
- *The map* $K \circ L$ *is a closure operator on* B.

Proof. We prove only the first assertion of the conclusion of the proposition. (Consider the other parts as exercises.) Note that $K(X) \subseteq (K \circ L)(K(X))$. Also note that $X \subseteq (L \circ K)(X)$ and hence $K(X) \supseteq K((L \circ K)(X))$. □

Associated with every Galois connection we have the following tasks:

- Determine the closed subsets of B with respect to the closure operator $K \circ L$.
- Determine the closed subsets of A with respect to the closure operator $L \circ K$.

Example: Let C be an algebraically closed field and let $I \subseteq C[X_1, \ldots, X_n]$ be an ideal. Then there is a subset S of C^n such that

$$I = \{f \in C[x_1, \ldots, x_n] : (\forall x \in S) f(x) = 0\}$$

iff and only if I is a radical ideal.

The following proposition can help with the tasks mentioned above. But the example shows that this general result does not usually provide a satisfactory solution.

Proposition 4. *Let* A *and* B *be sets and let* $K : \mathit{Boole}(A) \to \mathit{Boole}(B)$ *and* $L : \mathit{Boole}(B) \to \mathit{Boole}(A)$ *be maps which form a Galois connection. Then*

- *The subset* X *of* A *is closed iff* $(\exists Y \subseteq B) X = L(Y)$.
- *The subset* Y *of* B *is closed iff* $(\exists X \subseteq B) Y = L(X)$.

Proof. Use the last proposition. □