THE REYNOLD'S OPERATOR

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Notation and Review

In this lecture, we use K to denote a field with characteristic zero (e.g., $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$) and we use V to denote a vector space over K. We use $K[V] := \mathrm{Poly}(V \to K)$ to denote the K-valued polynomial functions on V.

Recall that a subset X of a finite dimensional vector space is a **variety** if it is the zero set of a set of polynomials on the vector space. The affine varieties together with polynomial maps (morphisms) $\alpha : X \to Y$ form a category which we are calling the **category of varieties**.

Associated with every affine variety X there is the **affine algebra** or **coordinate ring** $K[X] := \mathrm{Poly}(X \to K)$ of polynomial maps from X to the field K. The affine algebras together with algebra homomorphisms which fix constants form a category which we call the **category of affine algebras**.

There is a functor, called the **pullback functor**, from the category of varieties to the category of affine algebras; in particular, if $\alpha : X \to Y$ then the **pullback** of α is the map defined as follows:

$$\alpha^* : K[Y] \to K[X] : f \mapsto f \circ \alpha.$$

The category of varieties has products; in particular, if X and Y are varieties then we can describe the product set $X \times Y$ in terms of polynomials in a natural way. In connection with such products we have the following isomorphism in the category of affine algebras:

$$K[X] \otimes K[Y] \to K[X \times Y] : g \otimes h \mapsto ((x, y) \mapsto g(x)h(y)).$$

Recall that an **algebraic group** G is a variety together with morphisms (polynomial maps) $m : G \times G \to G$ and $i : G \to G$ which satisfy the axioms for a group. Recall that a **regular action of an algebraic group** G **on a variety** X is a morphism $\mu : G \times X \to G$ which satisfies the axioms for a group action.

Let V be a (not necessarily finite dimensional) vector space and let G be an algebraic group. Then a linear action $\mu : G \times V \to V$ of G on V is a **rational action** if there is a map $\nu : V \to V \otimes K[G]$ that satisfies the following condition for all $\sigma \in G$ and $v \in V$: If $\nu(v) = \sum_{i=1}^{l} v_i \otimes f_i$ then $\mu(\sigma, v) = \sum_{i=1}^{l} v_i f_i(\sigma)$.

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Let G be an algebraic group. We consider the dual vector space $K[G]^* := \mathrm{Lin}(K[G] \to K)$. The **convolution product** on $K[G]^*$ is defined as follows:

$$* := K[G]^* \otimes K[G]^* \to K[G]^* : (\gamma, \delta) \mapsto (\gamma \otimes \delta) \circ m^*$$

where m^* is the pullback of the multiplication map $m : G \times G \to G$. Recall that the space $K[G]^*$ with the convolution product is an asociative algebra with unit element. We call this algebra the **convolution algebra of G**.

Let $\mu : G \times V \to V$ be a rational representation. We extend this action to an action of $K[G]^*$ on V as follows. Let $\nu : V \to V \otimes K[G]$ be the map which determines rationality. We define

$$\cdot := K[G]^* \times V \to V : (\delta, v) \mapsto ((id \otimes \delta) \circ \nu)v,$$

that is, $\delta \cdot v := ((id \otimes \delta) \circ \nu)v$. This map is the **extended action**. Recall that this map gives V the structure of a $K[G]$-module.

Linearly Reductive Groups

Let G be a group and let $G \times V \to V$ be a representation. Recall that $V^G := \{v \in V : (\forall \sigma \in G)\ \sigma \cdot v = v\}$ denotes the space of fixed points. Also recall that the dual representation on $V^* := \mathrm{Lin}(V \to K)$ is defined by

$$G \times V^* \to V^* : (\sigma, f) \mapsto f \circ \sigma^{-1}.$$

Definition: Let G be a linear algebraic group. Then G is **linearly reductive** if, for every rational representation V and every nonzero v in V^G, there exists a linearly invariant function f in $(V^*)^G$ such that $f(v) \neq 0$.

Definition: Let G be an algebraic group and let X be a G-variety. Let $\mathcal{R} : K[X] \to K[X]^G$ be a linear map. Then $\mathcal{R}$ is a **Reynold's operator for the G-variety X** if it satisfies the following conditions:

- $(\forall f \in K[X]^G)\ \mathcal{R}(f) = f$
- $(\forall \sigma \in G)(\forall f \in K[X])\ \mathcal{R}(\sigma \cdot f) = \mathcal{R}(f)$.

Note that it follows that $\mathcal{R}^2 = \mathcal{R}$ and $\mathcal{R}$ is a projection of $K[V]$ onto $K[V]^G$. The second condition is expressed by saying that $\mathcal{R}$ is "G-invariant".

Proposition 1. *Let G be an algebraic group. Then the following conditions are equivalent.*

- *G is linearly reductive;*
- *for every rational representation V, there is a sub-representation $W \subseteq V$ such that $V = V^G \oplus W$ and $(W^*)^G = 0$;*
- *for every G-variety X there is a unique Reynold's operator $\mathcal{R}_X : K[X] \to K[X]^G$;*
- *for every rational representation V, and every sub-representation $W \subseteq V$, there is a sub-representation $Z \subseteq V$, such that $V = W \oplus Z$;*
- *for every rational representation V, there exist irreducible representations $V_1, V_2, \ldots, V_r \subseteq V$, such that $V = V_1 \oplus V_2 \oplus \cdots \oplus V_r$.*

Proof. ***TODO: Include a proof here.

□

Let G be an algebraic group. Note that, in the last result, we can take X to be G. That is, we can consider G to be a G-variety. Since G has only one orbit, in this case, $K[G]^G$ consists of only the constant functions on G and we can identify $K[G]^G$ with K. The next result shows that we can use the Reynold's operator $\mathcal{R}_G$ for the action $G \times G \to G$ to construct Reynold's operators for other actions.

Proposition 2. *Let G be an algebraic group. Let $\mathcal{R}_G : K[G] \to K$ be the Reynold's operator for the action $G \times G \to G$. If X is any affine G-variety then the Reynold's operator $\mathcal{R}_X$ for the action $G \times X \to X$ is given by the map $K[X] \to K[X]^G : f \mapsto \mathcal{R}_G \cdot f$.*

Remark: Let $\mu : G \times X \to X$ be the action map for G-variety X. We have the usual induced action of G on $K[X]$ which is defined by $\sigma \cdot f := f \circ \sigma^{-1}$. Let $\tilde{\mu} := X \times G \to X : (x, \sigma) \mapsto \sigma^{-1} \cdot x$. Recall that the comorphism $\tilde{\mu}^* : K[X] \to K[X] \otimes K[G]$ is a rationalizing map for the group action μ.

We have the extended action of $K[G]^*$ on $K[X]$ which is defined by:

$$\cdot := K[G]^* \times K[X] \to K[X] : (\delta, f) \mapsto ((id \otimes \delta) \circ \tilde{\mu}^*)f,$$

that is, $\delta \cdot f := ((id \otimes \delta) \circ \tilde{\mu}^*)f$. In other words, if $\tilde{\mu}^*(f) = \sum_i g_i \otimes h_i$ then $\delta \cdot f = \sum_i g_i \delta(h_i)$. Note that the Reynold's operator $\mathcal{R}_G$ is an element of the dual space $K[G]^*$. In the proposition, the dot in the expression $\mathcal{R}_G \cdot f$ represents this extended action. Taking δ to be $\mathcal{R}_G$, we get $\mathcal{R}_G \cdot f = \sum_i g_i \mathcal{R}_G(h_i)$.

Proof. Let $\mathcal{R} := K[X] \to K[X]^G : f \mapsto \mathcal{R}_G \cdot f$. We simply show that this map satisfies the properties defining the Reynold's operator for the group action $\mu : G \times X \to X$.

Claim: $(\forall f \in K[X]^G)\ \mathcal{R}(f) = f$.

Note $\tilde{\mu}^*(f) = f \otimes 1$ since $\tilde{\mu}^*(f)(x, \sigma) = f(\sigma^{-1} \cdot x) = f(x) = (f \otimes 1)(x, \sigma)$. Hence $\mathcal{R}(f) = \mathcal{R}_G \cdot f = f \mathcal{R}_G(1) = f$.

Claim: $\forall \sigma \in G)(\forall f \in K[X])\ \mathcal{R}(\sigma \cdot f) = \mathcal{R}(f)$

Since $\mathcal{R}_G$ is the Reynolds operator for G, we have

$$(\forall \sigma \in G)(\forall g \in K[G])\ \mathcal{R}_G(\sigma \cdot g) = \mathcal{R}_G(g).$$

In other words, $\mathcal{R}_G * \epsilon_\sigma = \mathcal{R}_G$. Hence

$$\mathcal{R}_G \cdot (\sigma \cdot f) = \mathcal{R} \cdot (\epsilon_\sigma \cdot f) = (\mathcal{R}_G * \epsilon_\sigma) \cdot f = \mathcal{R}_G \cdot f.$$

□