ALGEBRAIC GROUPS

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Notation

In this lecture, we use K to denote a field with characteristic zero (e.g., $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$) and we use V to denote a vector space over K (e.g., K^n). We use $K[V] := \text{Poly}(V \to K)$ to denote the K-valued polynomial functions on V. Note that when one picks a basis for V, the space $K[V]$ is isomorphic to the space of polynomials $K[x_1, \ldots, x_n]$ where n is the dimension of V.

Affine varieties

We briefly discuss some basic material on affine varieties. We need this background for our discussion of Reynold's operators.

Definition: For $F \subseteq K[V]$, we define the **zero set of** F or **solution set of** F, denoted $\text{Zero}(F)$, to be the set of elements of V which are solutions of every polynomial in F; in symbols,

$$\text{Zero}(F) := \{a \in V : (\forall f \in F)\ f(a) = 0\}.$$

Let X be a subset of V. If $X = \text{Zero}(F)$ for some subset F of $K[V]$, then X is an **(affine) variety** or **algebraic set**.

Definition: Let $X \subseteq K^m$ and $Y \subseteq K^n$ be varieties. A function $\phi : X \to Y$ is a **polynomial map** or **regular map** or **morphism** if there exist polynomials $f_1, \ldots, f_n$ in $K[K^m]$ such that, for all $a \in X$, $\phi(a) = (f_1(a), \ldots, f_n(a))$. We say that the n-tuple $f := (f_1, \ldots, f_n)$ **represents** the map ϕ. We shall use $\text{Poly}(X \to Y)$ to denote the set of all polynomial maps from X to Y.

Proposition 1. Affine category. *The collection of affine varieties together with polynomial maps forms a category.*

Proof. We leave the details to the reader. □

Definition: We call this category **the affine category** or **the category of varieties**.

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Affine algebras

There is another category that we need to discuss.

Definition: Let $X \subseteq V$ be an affine variety. We use $K[X]$ to denote the K-valued polynomial functions defined on X; in symbols,

$$K[X] := \text{Poly}[X \to K].$$

We view this set as a commutative algebra with identity. In particular, addition and multiplication are defined pointwise:

$$(f+g)(x) := f(x) + g(x)$$
$$(fg)(x) := f(x)g(x).$$

The constant function with value one is the identity. This algebra is called the **affine algebra of** X or the **coordinate ring of** X.

Definition: Let S be a subset of a vector space V. We use Ideal(S) to denote the set of all polynomials in $K[V]$ which vanish on all elements of S; in symbols,

$$\text{Ideal}(S) := \{f \in K[V] : (\forall a \in S)\ f(a) = 0\}.$$

It is easy to see that this set is an ideal in $K[V]$. We call it the **ideal determined by** S.

Proposition 2. Affine algebras as quotients of polynomial algebras. *Let $X \subseteq K^n$ be a variety. Then*

- *The polynomials f and g in $K[K^n]$ represent the same function on X iff $f - g \in Ideal(X)$.*
- *The algebras $K[X]$ and $K[K^n]/Ideal(X)$ are isomorphic.*

Proof. The proof is straight-forward. See, for example, Cox, Little and O'Shea(2007) for details. □

Definition: Let X and Y be varieties and let $\alpha : X \to Y$ be a polynomial map. Define the map α^* as follows:

$$\alpha^* := K[Y] \to K[X] : f \mapsto f \circ \alpha.$$

This map is the **pull-back map or comorphism determined by** α.

Proposition 3. Pull-back functor. *Let X, Y and Z be varieties and let $\alpha : X \to Y$ and $\beta : Y \to Z$ be polynomial maps. Then $(\beta \circ \alpha)^* = \alpha^* \circ \beta^*$ and $(Id_X)^* = Id_{K[X]}$.*

Proof. The proof is easy. We leave the details to the reader. □

The last proposition indicates that the pull-back map is a contra-variant functor on the category of varieties. The next proposition says more about this functor. (This proposition appears in Cox, Little and O'Shea(2007) .)

Proposition 4. Equivalence of categories. *Let X and Y be varieties.*

- *Let $\alpha : X \to Y$ be a polynomial map. Then $\alpha^* : K[Y] \to K[X]$ is an algebra homomorphism which fixes constants.*
- *Let $\phi : K[Y] \to K[X]$ be an algebra homomorphism which fixes constants. Then there is a unique polynomial map $\alpha : X \to Y$ such that $\phi = \alpha^*$.*

Proof. The proof of the first assertion is straight-forward. We leave the details to the reader.

Now let $\phi : K[Y] \to K[X]$ be a homomorphism of algebras which fixes constants. Assume that $Y \subseteq K^n$. Let $\pi_1, \ldots, \pi_n$ be the cooordinate function: $\pi_i := Y \to K : (y_1, \ldots, y_n) \mapsto y_i$. Note that $\pi_i \in K[Y]$ and hence $\phi(\pi_i) \in K[X]$. In other words, there exist $a_i \in K[X]$ such that $\phi(\pi_i) = a_i$. We consider the following polynomial map:

$$\alpha := X \to K^n : x \mapsto (a_1(x), \ldots, a_n(x)).$$

Claim: For all $p \in K[Y]$, the following equation holds in $K[X]$:

$$p \circ \alpha = \phi(p). \tag{*}$$

Since ϕ is a homomorphism, we have

$$\begin{aligned}(p \circ \alpha)(x) &= p(\phi(\pi_1)(x), \ldots, \phi(\pi_n)(x)) \\ &= p(\phi(\pi_1), \ldots, \phi(\pi_n))(x) \\ &= \phi(p(\pi_1, \ldots, \pi_n))(x).\end{aligned}$$

Claim: The range of α is a subset of Y; in symbols, $\alpha(X) \subseteq Y$.

We use the proposition which describes affine algebras as quotients. Let p be in Ideal(Y). Then $p = 0$ in $K[Y]$ and $\phi(p) = 0$ in $K[X]$. By the last claim, $p \circ \alpha = 0$ on X; in other words, for all $x \in X$, $p(\alpha(x)) = 0$.

***TODO: Perhaps add more detail to this paragraph.

Claim: The pull-back of α is ϕ: $\alpha^* = \phi$.

By equation $(*)$, we have, for all $p \in K[Y]$, $\alpha^*(p) = p \circ \alpha = \phi(p)$.

Claim: The mapping α is uniquely determined by ϕ.

Consider any $\beta : X \to Y$ which satisfies $\beta^* = \phi$. There exist polynomials $b_1, \ldots, b_n$ such that $(\forall x \in X)\ \beta(x) = (b_1(x), \ldots, b_n(x))$. Note $\beta^*(\pi_i) = \pi_i \circ \beta = b_i$. Similarly, $\alpha^*(\pi_i) = a_i$. Since $\alpha^* = \phi = \beta^*$, we have $a_i = b_i$ on X. Hence $\alpha = \beta$. □

Definition: We get the **category of affine algebras**: Its objects are the affine algebras $K[X]$ and its morphisms are the algebra homomorphisms $K[Y] \to K[X]$ that fix constants.

The last propositions says that, in some sense, the category of varieties and the category of affine algebras are equivalent.

We illustrate the use of this equivalence result to prove the following result.

Definition: Let A and B be objects in a category. A map $m : A \to B$ is a **monomorphism** if, for every pair of maps $x : C \to A$ and $y : C \to A$,

$m \circ x = m \circ y$ implies $x = y$. A map $p : A \to B$ is an epimorphism if, for every pair of maps $x : B \to C$ and $y : B \to C$, $x \circ p = y \circ p$ implies $x = y$.

Proposition 5. *Let $\alpha : X \to Y$ be a morphism in the category of varieties and let $\alpha^* : K[Y] \to K[X]$ be the corresponding pull-back morphism in the category of affine algebras.*

- *If α is a monomorphism then α^* is an epimorphism.*
- *If α is an epimorphism then α^* is a monomorphism.*

Proof. Consider two morphisms $\phi : K[Z] \to K[Y]$ and $\psi : K[Z] \to K[Y]$. Assume $\phi \circ \alpha^* = \psi \circ \alpha^*$. By the equivalence result we get morphisms $\beta : Y \to Z$ and $\gamma : Y \to Z$ such that $\phi = \beta^*$ and $\psi = \gamma^*$. Then

$$(\alpha \circ \beta)^* = \phi \circ \alpha^* = \psi \circ \alpha^* = (\alpha \circ \gamma)^*.$$

By the uniqueness part of the equivalence result we get $\alpha \circ \beta = \alpha \circ \gamma$. Hence $\beta = \gamma$ and $\phi = \beta^* = \gamma^* = \psi$.

The proof of the second part of proposition is similar (or follows by duality). □

Products of varieties

We want to see that the category of varieties has products. We also want to identify the affine algebra corresponding to such a product. We first recall the definition of product and co-product in a category. (We follow Freyd(1964) but the definition is standard; compare Lang(1993) , MacLane and Birkhoff(1979) , and/or Rotman(2002) .)

Definition: Let A, B and P be objects in a category. Then P is a **product of** A **and** B if there exist maps $P \to A$ and $P \to B$ such that, for every pair of maps $X \to A$ and $X \to B$, there is a unique map $X \to P$ such that $X \to A = X \to P \to A$ and $X \to B = X \to P \to B$. (The reader should draw the appropriate commuting diagram.)

Note that the product of two objects is uniquely determined up to isomorphism. Consequently, we shall use the phrase "the product" instead of "a product".

There is a dual notion.

Definition: Let A, B and C be objects in a category. Then C is a **co-product of** A **and** B if there exist maps $A \to C$ and $B \to C$ such that, for every pair of maps $A \to X$ and $B \to X$, there is a unique map $C \to X$ such that $A \to X = A \to C \to X$ and $B \to X = B \to C \to X$. (Again the reader should draw the appropriate commuting diagram.)

Note that the co-product of two objects is uniquely determined up to isomorphism. We shall use the phrase "the co-product" instead of "a co-product".

The following result indicates that we can use the direct product in the category of sets to construct the product of two varieties. (The first part of this result appears in Eisenbud(1994) .)

Proposition 6. Products of varieties. *Let $X \subseteq K^m$ and $Y \subseteq K^n$ be varieties.*

- *The product set $X \times Y \subseteq K^{m+n}$ is a variety. In particular, $X \times Y = Z$ where $Z := \mathrm{Zero}\{f(x) + g(y) : f \in \mathrm{Ideal}(X) \wedge g \in \mathrm{Ideal}(Y)\}$.*
- *The variety $X \times Y$ is the product of X and Y is the affine category.*

Proof. Claim: $X \times Y = Z$.

$\subseteq$: Consider any $(a, b) \in X \times Y$. If $f \in \mathrm{Ideal}(X)$ and $g \in \mathrm{Ideal}(Y)$ then $(f(x) + g(y))(a, b) = f(a) + g(b) = 0$.

$\supseteq$: Suppose $(a, b) \notin X \times Y$. Then either $a \notin X$ or $b \notin Y$. Suppose $a \notin X$. Then there is a polynomial function $f \in K[K^m]$ such that $f(a) \neq 0$. Take $g = 0$. Hence $(a, b) \notin Z$. The other case, $b \notin Y$, can be treated in a similar way.

Claim: : The variety $X \times Y$ is a product of X and Y in the affine category.

Let W be a variety and let $\alpha : W \to X$ and $\beta : W \to Y$ be morphisms. Consider the following map:

$$W \to X \times Y : u \mapsto (\alpha(u), \beta(u)).$$

This is the unique map in the category of sets determined by the maps α and β. But this map is clearly a polynomial map. □

It follows at once (by duality) that the category of afine algebras has coproducts. The following result is more specific. (This result appears in Humphreys(1981) .)

Proposition 7. Coproducts of affine algebras. *The category of affine algebras has coproducts. In particular, if X and Y are affine varieties the tensor product $K[X] \otimes K[Y]$ is isomorphic to $K[X \times Y]$; in symbols,*

$$K[X] \otimes K[Y] \simeq K[X \times Y].$$

This tensor product is the coproduct of $K[X]$ and $K[Y]$.

Proof. Let $X \subseteq K^m$ and $Y \subseteq K^n$. As in the proof of the result concerning products of varieties, we regard $K[K^m] = K[x]$ where $x := (x_1, \ldots, x_m)$, $K[K^n] = K[y]$ where $y := (y_1, \ldots, y_n)$ and $K[K^{m+n}] = K[x, y]$ where $(x, y) := (x_1, \ldots, x_m, y_1, \ldots, y_n)$. Consider the following map:

$$\sigma := K[X] \times K[Y] \to K[X \times Y] : (g, h) \mapsto ((x, y) \mapsto g(x)h(y)).$$

Note that this map is bilinear. We use the universal mapping property of the tensor product to get a unique map

$$\hat{\sigma} := K[X] \otimes K[Y] \to K[X \times Y] : g \otimes h \mapsto ((x, y) \mapsto g(x)h(y))$$

corresponding to σ.

Claim: The map $\hat{\sigma}$ is surjective.

Note that we can express every polynomial in $x_1, \ldots, x_m, y_1, \ldots, y_n$ as a finite sum of products $g(x)h(y)$.

Claim: The map $\hat{\sigma}$ is injective.

Suppose not. Then there is an nonzero element f of $K[X] \otimes K[Y]$ such that $\hat{\sigma}(f) = 0$. Let $f = \sum_{i=1}^{r} g_i \otimes h_i$ where the g_i are in $K[X]$ and the h_i are in $K[Y]$. We may assume that r is minimal. Since f is nonzero there exists an index j such that h_j is nonzero. Hence there is an element b in Y such that $h_j(b)$ is nonzero. Since $\hat{\sigma}(f) = 0$, we have $\sum_{i=1}^{l} c_i g_i(x) = 0$ where $c_i := h_i(b)$ for $i = 1, \ldots, r$. In other words, $\sum_{i=1}^{r} c_i g_i = 0$ in $K[X]$. It follows that $r = 1$ since otherwise we could reduce r.

We now repeat the argument with $f = g \otimes h$. Again $\hat{\sigma}(f) = 0$ means that, for all $x \in X$ and $y \in Y$, $g(x)h(y) = 0$. If f is nonzero then h is nonzero and $h(b)$ is nonzero for some $b \in Y$. But then, for all $x \in X$, $cg(x) = 0$ where $c := g(b)$. Thus $g = 0$ in $K[X]$. □

Algebraic Groups

Here we follow the appendix on "Linear Algebraic Groups" in Derksen and Kemper(2002) . We omit some of the proofs that can be found there.

Definition: A **(linear) algebraic group** is an (affine algebraic) variety G together with a unit element $e \in G$ and morphisms $m : G \times G \to G$ and $i : G \to G$ which satisfy the following conditions:

- $(\forall \sigma \in G)\ m(\sigma, e) = m(e, \sigma) = \sigma$
- $(\forall \sigma \in G)\ m(\sigma, i(\sigma)) = m(i(\sigma), \sigma) = e$
- $(\forall \alpha, \beta, \gamma \in G)\ m(\alpha, m(\beta, \gamma)) = m(m(\alpha, \beta), \gamma))$.

We often write $\sigma\tau$ in place of $m(\sigma, \tau)$ and σ^{-1} in place of $i(\sigma)$.

The following result appears in Humphreys(1981) and Borel(1991): Every linear algebraic group is isomorphic to a Zariski closed subgroup of some general linear group. This justifies the use of the word "linear" in the phrase "linear algebraic group".

The following result shows that we can regard G as a subset of the dual space $K[G]^* := \mathrm{Lin}(K[G] \to K)$.

Proposition 8. Standard injection. *The following map is injective:*

$$\epsilon := G \to K[G]^* : \sigma \mapsto (f \mapsto f(\sigma)).$$

Proof. Let $G \subseteq V$. We need to see that if σ is different than e then there is a linear map $f : K[G] \to K$ such that $f(\sigma)$ is different than $f(e)$.

***TODO: Finish this proof.

□

Note that, associated with an algebraic group G, we have the following comorphisms: $m^* : K[G] \to K[G] \otimes K[G]$ and $i^* : K[G] \to K[G]$. We can state the axioms for an algebraic group in terms of these comorphisms where $\epsilon := \epsilon(e)$:

- $(\mathrm{id} \otimes \epsilon) \circ m^* = (\epsilon \otimes \mathrm{id}) \circ m^* = \mathrm{id}$
- $(i^* \otimes \mathrm{id}) \circ m^* = (\mathrm{id} \otimes i^*) \circ m^* = \epsilon$
- $(\mathrm{id} \otimes m^*) \circ m^* = (m^* \otimes \mathrm{id}) \circ m^*$.

Definition: Let G be an algebraic group and let X be a variety. Then a **regular action of** G **on** X is a morphism $\mu : G \times X \to X$ that satisfies the following conditions:

- $(\forall x \in X)\ \mu(e, x) = x$
- $(\forall x \in X)(\forall \sigma, \tau \in G)\ \mu(\sigma, \mu(\tau, x)) = \mu(\sigma\tau, x)$.

We often write $\sigma \cdot x$ in place of $\mu(\sigma, x)$. Instead of saying the G acts regularly on X, we often simply say that X is a G**-variety**.

Definition: Let V be a (not necessarily finite dimensional) vector space and let G be an algebraic group. Then a linear action $\mu : G \times V \to V$ of G on V is a **rational action** if there is a map $\nu : V \to V \otimes K[G]$ that satisfies the following condition for all $\sigma \in G$ and $v \in V$: If $\nu(v) = \sum_{i=1}^{l} v_i \otimes f_i$ then $\mu(\sigma, v) = \sum_{i=1}^{l} v_i f_i(\sigma)$. We call the map ν a **rationalizing map for the action** μ.

The following two propositions provide examples of rational actions.

Proposition 9. *Let G be an algebraic group, let V be a vector space and let $\mu : G \times V \to V$ be a linear action. If the dimension of V is finite then this action is rational.*

Proof. Let $b_1, \dots, b_n$ be a basis for V. For σ in G let $f_{ij}(\sigma)$ be the matrix of σ with respect to the basis; in other words, $f_{ij}(\sigma)$ is defined by $\sigma \cdot b_j = \sum_i b_i f_{ij}(\sigma)$. We define the map $\nu : V \to V \otimes K[G]$ by setting $\nu(b_j) := \sum_i b_i \otimes f_{ij}$ and extending linearly. In particular, if $v = \sum_j c_j b_j$, then $\nu(v) := \sum_i b_i \otimes (\sum_j c_j f_{ij})$. We then have

$$\sigma \cdot v = \sigma \cdot (\sum_j c_j b_j) = \sum_j c_j(\sigma \cdot b_j) = \sum_j c_j(\sum_i b_i f_{ij}(\sigma)) = \sum_i \sum_j b_i c_j f_{ij}(\sigma).$$

□

Proposition 10. *Let G be an algebraic group. Let X be a G-variety. Then the induced representation on $K[X]$ is rational; in other words, the following representation is rational:*

$$\cdot := G \times K[X] \to K[X] : (\sigma, f) \mapsto (x \mapsto f(\sigma^{-1} \cdot x)).$$

Proof. Let $\mu : G \times X \to X$ be the morphism of the G action on X. Define

$$\tilde{\mu} := X \times G \to X : (x, \sigma) \mapsto \sigma^{-1} \cdot x.$$

Note that $\tilde{\mu}$ is a morphism. We consider the corresponding comorphism:

$$\tilde{\mu}^* : K[X] \to K[X] \otimes K[G] \simeq K[X \times G].$$

Claim: The map $\tilde{\mu}^*$ is a rationalizing map for μ.

For f in $K[X]$, let $\tilde{\mu}^*(f) = \sum g_i \otimes h_i$ where $g_i \in K[X]$ and $h_i \in K[G]$. Consider any $\sigma \in G$ and $x \in X$. We have

$$\tilde{\mu}^*(f)(x, \sigma) = (f \circ \tilde{\mu})(x, \sigma) = f(\sigma^{-1} \cdot x) = (\sigma \cdot f)(x).$$

We also have

$$\tilde{\mu}^*(f)(x,\sigma) = \sum_i (g_i \otimes h_i)(x,\sigma) = \sum_i g_i(x)h_i(\sigma).$$

Hence $\sigma \cdot f = \sum_i g_i h_i(\sigma)$. □

Proposition 11. *Let $\mu : G \times V \to V$ be a regular action. This action is rational iff there is a map $\nu : V \to V \otimes K[V]$ such that, for all $\sigma \in G$ and $v \in V$,*

$$\mu(\sigma, v) = ((id \otimes \epsilon(\sigma)) \circ \nu)v.$$

Proof. Assume that ν exists satisfying the condition in the definition of rationial. Consider any $\sigma \in G$ and $v \in V$. Let $\nu(v) = \sum v_i \otimes f_i$. Then

$$((\mathrm{id} \otimes \epsilon(\sigma)) \circ \nu)(v) = (\mathrm{id} \otimes \epsilon(\sigma)) \sum v_i \otimes f_i = v_i f_i(\sigma) = \mu(\sigma, v).$$

The proof of the other implication is similar. □

The following result shows that a rational action is "locally finite".

Proposition 12. Local finiteness. *Let G be an algebraic group and let $G \times V \to V$ be a rational linear action. Then, for every $v \in V$, the span of the orbit of v, namely $W := \mathrm{Span}(G{\cdot}v)$, has finite dimension and is G-stable. The restriction of the G-action to W, $G \times W \to W$, is also rational.*

Proof. Let $\nu(v) = \sum_{i=1}^{l} v_i \otimes f_i$. Let $W := \mathrm{Span}(G \cdot v)$ be the linear span of the orbit of v.

Claim: $W \subseteq \mathrm{Span}(v_1, \ldots, v_l)$

We have, for all $\sigma \in G$,

$$\sigma \cdot v = \sum_{i=1}^{l} v_i f_i(\sigma).$$

Thus $G \cdot v \subseteq \mathrm{Span}(v_1, \ldots, v_l)$ and hence $\mathrm{Span}(G \cdot v) \subseteq \mathrm{Span}(v_1, \ldots, v_l)$.

Claim: The subspace W is G-stable.

Note that there exist $\sigma_1, \ldots, \sigma_m$, such that $W = \mathrm{Span}(\sigma_1 \cdot v, \ldots, \sigma_m \cdot v)$. We then have, for all $\sigma \in G$,

$$\sigma \cdot (\sum_{i=1}^{m} c_i \sigma_i \cdot v) = \sum_{i=1}^{m} c_i (\sigma\sigma_i) \cdot v.$$

Claim: The restriction of the G action to W is rational.

Use the restriction of ν to W.

□

The Lie algebra of an algebraic group

Again we follow the appendix on "Linear Algebraic Groups" in Derksen and Kemper(2002) and we omit some of the proofs that can be found there.

Let G be an algebraic group. We continue to consider the dual vector space $K[G]^* := \mathrm{Lin}(K[G] \to K)$.

Definition: The **convolution product** on $K[G]^*$ is defined as follows:

$$* := K[G]^* \otimes K[G]^* \to K[G]^* : (\gamma, \delta) \mapsto (\gamma \otimes \delta) \circ m^*$$

where m^* is the pullback of the multiplication map $m : G \times G \to G$. In other words, if $f \in K[G]^*$ and $m^*(f) = \sum_{i=1}^{l} g_i \otimes h_i$ then

$$(\gamma * \delta)(f) = (\gamma \otimes \delta)(\sum_{i=1}^{l} g_i \otimes h_i) = \sum_{i=1}^{l} \gamma(g_i)\delta(h_i).$$

We call the set $K[G]^*$ together with this product the **convolution algebra** of G. The next proposition justifies this terminology.

Proposition 13. Convolution algebra. *Let G be an algebraic group. The space $K[G]^*$ with the convolution product is an asociative algebra with unit element.*

Proof. ***TODO: Include a proof here.

□

Recall that in any associative algebra, we can define a Lie bracket operation. In particular, we have a **Lie bracket in** $K[G]^*$ defined as follows:

$$[,] : K[G]^* \times K[G]^* \to K[G]^* : (\gamma, \delta) \mapsto \gamma * \delta - \delta * \gamma.$$

In other words, $[\gamma, \delta] := \gamma * \delta - \delta * \gamma$. With this operation $K[G]^*$ is a Lie algebra. However, there is sub-algebra of this Lie algebra which is called "the Lie algebra of G".

Definition: Let G be an algebraic group. Define the subset $\mathrm{Lie}(G)$ of $K[G]^*$ as follows:

$$\mathrm{Lie}(G) := \{\delta \in K[G]^* : (\forall f, g \in K[G]) \ \delta(fg) = \delta(f)g(e) + f(e)\delta(g)\}.$$

This set is called **the Lie algebra of** G.

The following proposition justifies this terminology.

Proposition 14. The Lie algebra. *The set* $\mathrm{Lie}(G)$ *with the Lie bracket is a Lie algebra.*

Proof. Note that $Lie(G)$ is a subspace. So we only need to see that this set is closed under the Lie bracket.

***TODO: Include a proof here.

□

Definition: Let $\mu : G \times V \to V$ be a rational representation. We want to extend this action to an action of $K[G]^*$ on V. Let $\nu : V \to V \otimes K[G]$ be the map which determines rationality. We define

$$\cdot := K[G]^* \times V \to V : (\delta, v) \mapsto ((id \otimes \delta) \circ \nu)v,$$

that is, $\delta \cdot v := ((id \otimes \delta) \circ \nu)v$. In other words, if $\nu(v) = \sum v_i \otimes h_i$ then $\delta \cdot v = \sum v_i \delta(h_i)$. We call this map the **extended action**. The next two propositions justify this terminology.

Recall that the map ϵ is a injection of G into $K[G]^*$. The next proposition shows that the action of an element of G on an element of V is the same as the action of the corresponding element of $K[G]^*$ on that element of V. Thus we are extending the the action of G.

Proposition 15. Extension. *Let $\mu : G \times V \to V$ be a rational representation. Then for all $\sigma \in G$ and $v \in V$, $\sigma \cdot v = \epsilon(\sigma) \cdot v$.*

Proof. Let ν be a rationalizing map for μ. Recall that if $\nu(v) = \sum v_i \otimes f_i$ then $\sigma \cdot v = \sum v_i f_i(\sigma)$. We also have

$$\epsilon(\sigma) \cdot v = (\mathrm{id} \otimes \epsilon(\sigma)) \sum v_i \otimes f_i = \sum_i v_i f_i(\sigma).$$

□

Proposition 16. Extended action. *The map $K[G]^* \times V \to V$ gives V the structure of a $K[G]$-module.*