Quantum Reversibility and Echoes in Interacting Systems

C. Petitjean[1] and Ph. Jacquod[2]
[1] *Département de Physique Théorique, Université de Genève, CH-1211 Genève 4, Switzerland*
[2] *Physics Department, University of Arizona, 1118 E. 4th Street, Tucson, AZ 85721, USA*
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In Echo experiments, imperfect time-reversal operations are performed on a subset of the total number of degrees of freedom. To capture the physics of these experiments, we introduce a partial fidelity $\mathcal{M}_{\rm B}(t)$, the Boltzmann echo, where only part of the system's degrees of freedom can be time-reversed. We present a semiclassical calculation of $\mathcal{M}_{\rm B}(t)$. We show that, as the time-reversal operation is performed more and more accurately, the decay rate of $\mathcal{M}_{\rm B}(t)$ saturates at a value given by the decoherence rate of the controlled degrees of freedom due to their coupling to uncontrolled ones. We connect these results with NMR spin echo experiments.

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One of the central problems faced by the founders of statistical physics in the last decades of the nineteenth century was to reconcile the time-asymmetric evolution of macroscopic systems with time-symmetric microscopic dynamics [1]. They came up with a probabilistic solution to this *irreversibility paradox*. Macroscopic states, they argued, are superpositions of an enormous amount of microscopic states, the majority of them evolving in accordance with the second law of thermodynamics. The likelihood that a macroscopic state violates the second law of thermodynamics is thus minute, typically exponentially small in the number of atoms it contains. Irreversibility at the macroscopic level follows "by assuming a very improbable (i.e. with a very low entropy) initial state of the entire universe" [2, 3]. This mechanism works equally well in either quantum or classical systems.

Simple mechanisms of irreversibility already exist at the microscopic level in chaotic (in particular mixing) classical systems with few degrees of freedom. As a matter of fact, mixing ensures that, after a sufficiently long evolution time, two initially well separated phase-space distributions will evenly fill phase-space cells of any given size. Since phase-space points can never be located with infinite precision, irreversibility sets in after mixing has occurred on a scale smaller than the phase-space resolution scale. This mechanism cannot be carried over to quantum systems, however, mostly because the Schrödinger time-evolution is unitary, in either real- or momentum-space. Microscopic quantum systems are generically stable under time-reversal, even when their classical counterpart is irreversible [4]. Peres instead suggested to investigate quantum irreversibility at the microscopic level through the fidelity

$$\mathcal{M}_{\rm L}(t) = \left|\langle\psi_0 \left|\exp[iHt]\exp[-iH_0t]\right|\psi_0\rangle\right|^2 , \qquad (1)$$

with which a quantum state ψ_0 can be reconstructed by inverting the dynamics after a time t with a perturbed Hamiltonian $H = H_0 + \Sigma$ [5]. Because of its connection with the gedanken time-reversal experiment proposed by Loschmidt in his argument against Boltzman's H-theorem [1], $\mathcal{M}_{\rm L}(t)$ has been dubbed the *Loschmidt Echo* by Jalabert and Pastawski [6].

Echo experiments abound in nuclear magnetic resonance [7, 8], optics [9], atomic [10], and condensed matter physics [11]. Fundamentally, they are all based on the same principle of a sequence of electromagnetic pulses whose purpose it is to reverse the sign of the Hamiltonian, $H_0 \to -H_0$, by means of effective changes of coordinate axes [7]. Imperfections in the pulse sequence result instead in $H_0 \to -H_0 - \Sigma$, and one therefore expects the Loschmidt Echo to capture the physics of the experiments. This line of reasoning however neglects the fact that the time-reversal operation affects at best only part of the system, for instance because the system is composed of so many degrees of freedom, that the time arrow can be inverted only for a fraction of them. This is generically the case, as any system is coupled to an external, uncontrolled environment. To capture the physics of echo experiments one thus has to take into account that (i) the system decomposes into two interacting subsystems 1 and 2; (ii) the initial state of the controlled subsystem 1 is prepared, i.e. well defined, and its final state is measured and compared to the initial one; (iii) both the initial and final states of the uncontrolled subsystem 2 are unknown; (iv) the Hamiltonian of system 1 is time-reversed with some tunable accuracy, however both the Hamiltonian of system 2 and the interaction between the two subsystems are uncontrolled. We therefore propose to investigate the physics of echo experiments by means of the following partial fidelity (we set $\hbar \equiv 1$)

$$\mathcal{M}_{\rm B}(t) = \Big\langle \langle\psi_1|{\rm Tr}_2\left[e^{-i\mathcal{H}_{\rm b}t}e^{-i\mathcal{H}_{\rm f}t}\rho_0 e^{i\mathcal{H}_{\rm f}t}e^{i\mathcal{H}_{\rm b}t}\right]|\psi_1\rangle \Big\rangle, \qquad (2)$$

where the forward and backward (partially time-reversed) Hamiltonians read

$$\mathcal{H}_{\rm f} = H_1 \otimes I_2 + I_1 \otimes H_2 + \mathcal{U}_{\rm f}, \qquad (3a)$$
$$\mathcal{H}_{\rm b} = -[H_1 + \Sigma_1] \otimes I_2 + I_1 \otimes [H_2 + \Sigma_2] + \mathcal{U}_{\rm b}. \qquad (3b)$$

The experiment starts with an initial density matrix $\rho_0 = |\psi_1\rangle\langle\psi_1| \otimes \rho_2$, which is propagated forward in time with $\mathcal{H}_{\rm f}$. After a time t, we invert the dynamics of system 1. The imperfection in that time-reversal operation is modelled by Σ_1, while Σ_2 allows for system 2 to be affected by this operation (we will see below that tracing over the degrees of freedom of system 2 makes $\mathcal{M}_{\rm B}(t)$ independent of either H_2 or Σ_2). We leave open the possibility that the interaction between the two systems is affected by the time-reversal operation, i.e. $\mathcal{U}_{\rm f}$ may or may not be equal to $\mathcal{U}_{\rm b}$. Because one has no control over system 2, the corresponding degrees of freedom are traced out. For the same reason, the outmost brackets in Eq. (2) indicate an average over ρ_2. We name $\mathcal{M}_{\rm B}(t)$ the *Boltzmann echo* to stress its connection to Boltzmann's counterargument to Loschmidt that time cannot be inverted for all components of a system with many degrees of freedom.

In this article, we present a semiclassical calculation of the Boltzmann echo for two classically chaotic subsystems along the lines of Refs.[6, 13, 14], and compare our results with those obtained from a Random Matrix Theory (RMT) treatment of the problem. Our main result is that, in the regime of classically weak but quantum mechanically strong imperfection Σ_1 and coupling $\mathcal{U}_{\rm f,b}$, $\mathcal{M}_{\rm B}(t)$ is the sum of two exponentials

$$\mathcal{M}_{\rm B}(t) \simeq \exp\left[-\left(\Gamma_{\Sigma_1}+\Gamma_{\rm f}+\Gamma_{\rm b}\right)t\right]+\alpha_1\exp\left[-\lambda_1 t\right]. \quad (4)$$

Here, α_1 is a weakly time-dependent prefactor, λ_1 is the classical Lyapunov exponent of system 1, and Γ_{Σ_1} and $\Gamma_{\rm f,b}$ are given by classical correlators for Σ_1 and $\mathcal{U}_{\rm f,b}$ respectively (see below). Equivalently, they can be regarded as the golden rule width of the Lorentzian broadening of the levels of H_1 induced by Σ_1 and $\mathcal{U}_{\rm f,b}$ respectively [12]. Together with the one- and two-particle level spacings $\delta_{1,2}$ and bandwidths $B_{1,2}$, they define the range of validity of the semiclassical approach as $\delta_1 < \Gamma_{\Sigma_1} < B_1$, $\delta_2 < \Gamma_{\rm f,b} < B_2$ [12, 13, 14]. The second term on the right-hand side of Eq. (4) exists exclusively for a classically meaningful initial state ψ_1 such as a Gaussian wavepacket or a position state, but the first term is much more generic. It emerges from both a semiclassical or a RMT treatment and does not depend on the initial preparation ψ_1 of system 1. Other regimes of decay exist, which we here mention for the sake of completeness. For quantum mechanically weak $\Gamma_{\Sigma_1} \ll \delta_1$ and $\Gamma_{\rm f,b} \ll \delta_2$, one has a Gaussian decay,

$$\mathcal{M}_{\rm B}(t) = \exp\left[-\left(\overline{\Sigma_1^2}/4+\overline{\mathcal{U}_{\rm f}^2}/2+\overline{\mathcal{U}_{\rm b}^2}/2\right)t^2\right], \quad (5)$$

in term of the typical squared matrix elements of Σ_1 and $\mathcal{U}_{\rm f,b}$. Also, at short times a parabolic decay of $\mathcal{M}_{\rm B}(t)$ prevails for any coupling strength. Finally, if system 1 is integrable, the decay of $\mathcal{M}_{\rm B}(t)$ is power-law in time.

The equivalence between Boltzmann and Loschmidt echoes is broken by $\Gamma_{\rm f,b}$, the decoherence rate of system 1 induced by the coupling to system 2 (or by $\overline{\mathcal{U}_{\rm f,b}^2}$ at weak interaction). Skillfull experimentalists can thus investigate decoherence in echo experiments with weak time-reversal imperfection Σ_1 for which $\Gamma_{\Sigma_1} \ll \Gamma_{\rm f,b}$, and thus $\mathcal{M}_{\rm B}(t) \simeq \exp[-(\Gamma_{\rm f}+\Gamma_{\rm b})t]$ (or $\mathcal{M}_{\rm B}(t) \simeq \exp[-(\overline{\mathcal{U}_{\rm f}^2}+\overline{\mathcal{U}_{\rm b}^2})\ t^2/2]$ at weak interaction) as Σ_1 is reduced. This might well be the explanation for the experimentally observed Σ_1-independent decay of polarization echoes [15].

We now present our calculation. As starting point, we take chaotic one-particle Hamiltonians $H_{1,2}$, and a smooth interaction potential $\mathcal{U}$ which depends only on the distance between the particles. We assume that it is characterized by a typical classical length scale, which in particular is larger than the de Broglie wavelength σ of particle 1. For pedagogical reasons, we take narrow Gaussian wavepackets for the initial state of both particles, $\psi_i(\mathbf{q}) = \langle \mathbf{q}|\psi_i\rangle = (\pi\sigma^2)^{-d_i/4}\exp[i\mathbf{p}_i\cdot(\mathbf{q}-\mathbf{r}_i)-|\mathbf{q}-\mathbf{r}_i|^2/2\sigma^2]$. We note however that within our semiclassical approach, more general states can be taken for the uncontrolled system 2, such as random pure states $\rho_2 = \sum_{\alpha\beta} a_\alpha a_\beta^* |\phi_\alpha\rangle\langle\phi_\beta|$, random mixtures $\rho_2 = \sum_\alpha |a_\alpha|^2 |\phi_\alpha\rangle\langle\phi_\alpha|$ or thermal mixtures $\rho_2 = \sum_n \exp[-\beta E_n]\,|n\rangle\langle n|$. Arbitrary initial states for both subsystems can be considered within the RMT approach.

From Eqs. (2) and (3) we can rewrite $\mathcal{M}_{\rm B}(t)$ as

$$\mathcal{M}_{\rm B}(t) \;=\; \int {\rm d}\mathbf{z}_2 \;\left|\int \prod_{i=1}^{2}{\rm d}\mathbf{x}_i \prod_{j=1}^{3}{\rm d}\mathbf{q}_j\;\psi_1(\mathbf{q}_1)\psi_2(\mathbf{q}_2)\psi_1^\dagger(\mathbf{q}_3)\left\langle \mathbf{q}_3,\mathbf{z}_2\left|e^{-i\mathcal{H}_{\rm b}t}\right|\mathbf{x}_1,\mathbf{x}_2\right\rangle\left\langle \mathbf{x}_1,\mathbf{x}_2\left|e^{-i\mathcal{H}_{\rm f}t}\right|\mathbf{q}_1,\mathbf{q}_2\right\rangle\right|^2. \quad (6)$$

We next introduce the semiclassical propagators ($a = {\rm f},{\rm b}$ labels forward or backward evolution; $\epsilon^{(f)} = -\epsilon^{(b)} = 1$),

$$\left\langle \mathbf{x}_1,\mathbf{x}_2\left|e^{-i\mathcal{H}_a t}\right|\mathbf{q}_1,\mathbf{q}_2\right\rangle \;=\; \sum_{s_1,\,s_2}\mathcal{C}_{s_1,s_2}^{1/2}\exp\left[i\left\{\epsilon^{(a)}S_{s_1}^{(a)}(\mathbf{q}_1,\mathbf{x}_1;t)+S_{s_2}^{(a)}(\mathbf{q}_2,\mathbf{x}_2;t)+\mathcal{S}_{s_1,s_2}^{(a)}(\mathbf{q}_1,\mathbf{x}_1;\mathbf{q}_2,\mathbf{x}_2;t)\right\}\right], \quad (7)$$

which are expressed as sums over pairs of classical trajectories, labeled s_i (l_i) for particle i connecting $\mathbf{q}_i$ to $\mathbf{x}_i$ in the time t with dynamics determined by H_i or $H_i+\Sigma_i$. Under our assumption of a classically weak coupling, classical trajectories are only determined by the one-particle Hamiltonians. Each pair of paths gives a contribution containing one-particle action integrals denoted by S_{s_i} (where we included the Maslov indices) and two-particle action integrals $\mathcal{S}_{s_1,s_2}^{(\rm f,b)} = \int_0^t {\rm d}\tau\,\mathcal{U}_{\rm f,b}[\mathbf{q}_{s_1}(\tau),\mathbf{q}_{s_2}(\tau)]$ accumulated along s_1 and s_2 and the determinant $\mathcal{C}_{s_1,s_2} = C_{s_1}C_{s_2}$ of the stability matrix corresponding to the two-particle dynamics in the $(d_1+d_2)-$dimensional space [16].

Our choice of initial Gaussian wave packets allows us to linearize the one-particle action integrals in $\mathbf{q}_j - \mathbf{r}_i$. We furthermore set $\mathcal{S}^{(a)}_{s_1,s_2}(\mathbf{q}_1, \mathbf{x}_1; \mathbf{q}_2, \mathbf{x}_2; t) \simeq \mathcal{S}^{(a)}_{s_1,s_2}(\mathbf{r}_1, \mathbf{x}_1; \mathbf{r}_2, \mathbf{x}_2; t)$, keeping in mind that $\mathbf{r}_1$ and $\mathbf{r}_2$, taken as arguments of the two-particle action integrals, have an uncertainty $\mathcal{O}(\sigma)$. We then perform six Gaussian integrations to get

$$\mathcal{M}_{\rm B}(t) \;=\; (4\pi\sigma^2)^{\frac{2d_1+d_2}{2}} \int \prod_{i=1}^{2} \mathrm{d}\mathbf{x}_i \mathrm{d}\mathbf{y}_i \mathrm{d}\mathbf{z}_2 \sum_{\rm paths} \mathcal{A}_{s_1}\mathcal{A}_{s_2}\mathcal{A}^\dagger_{s_3}\mathcal{A}^\dagger_{s_4}\mathcal{A}^\dagger_{l_1}\mathcal{A}_{l_3}C^{\frac{1}{2}}_{l_2}C^{\frac{1}{2}\dagger}_{l_4} \exp\left[i\left\{\Phi_1+\Phi_2+\Phi_{12}\right\}\right], \tag{8}$$

where we wrote $\mathcal{A}_{s_i} = C^{\frac{1}{2}}_{s_i}\exp[-\frac{\sigma^2}{2}(\mathbf{p}_{s_i}-\mathbf{p}_i)^2]$. Paths with odd (even) indices correspond to system 1 (2). The semiclassical expression for $\mathcal{M}_{\rm B}(t)$ is obtained by enforcing a stationary phase condition on Eq. (8), i.e. keeping only terms which minimize the variation of the three action phases

$$\begin{aligned}\Phi_1 \;&=\; S^{\rm (f)}_{s_1}(\mathbf{r}_1,\mathbf{x}_1;t) - S^{\rm (b)}_{l_1}(\mathbf{r}_1,\mathbf{x}_1;t)\\ &\quad -S^{\rm (f)}_{s_3}(\mathbf{r}_1,\mathbf{y}_1,t) + S^{\rm (b)}_{l_3}(\mathbf{r}_1,\mathbf{y}_1;t), \qquad (9a)\\ \Phi_2 \;&=\; S^{\rm (f)}_{s_2}(\mathbf{r}_2,\mathbf{x}_2;t) + S^{\rm (b)}_{l_2}(\mathbf{x}_2,\mathbf{z}_2;t)\\ &\quad -S^{\rm (f)}_{s_4}(\mathbf{r}_2,\mathbf{y}_2;t) - S^{\rm (b)}_{l_4}(\mathbf{y}_2,\mathbf{z}_2;t), \qquad (9b)\\ \Phi_{12} \;&=\; \mathcal{S}^{\rm (f)}_{s_1,s_2} + \mathcal{S}^{\rm (b)}_{l_1,l_2} - \mathcal{S}^{\rm (f)}_{s_3,s_4} - \mathcal{S}^{\rm (b)}_{l_3,l_4}. \qquad (9c)\end{aligned}$$

The semiclassically relevant terms are identified by path contractions. The first stationary phase approximation over Φ_1 corresponds to contracting unperturbed paths with perturbed ones, $s_1 \simeq l_1$ and $s_3 \simeq l_3$. This pairing is allowed by our assumption of a classically weak Σ_1 [17]. The phase Φ_1 is then given by the difference of action integrals of the perturbation Σ_1 on paths s_1 and s_3, $\Phi_1 = \delta S_{s_1}(\mathbf{r}_1,\mathbf{x}_1;t) - \delta S_{s_3}(\mathbf{r}_1,\mathbf{y}_1,t)$, with $\delta S_{s_i} = \int_0^t \mathrm{d}\tau \Sigma_1[\mathbf{q}_{s_i}(\tau)]$. Here, $\mathbf{q}_{s_i}(\tau)$ lies on s_i with $\mathbf{q}_{s_i}(0) = \mathbf{r}_1$ and $\mathbf{q}_{s_1}(t) = \mathbf{x}_1$, $\mathbf{q}_{s_3}(t) = \mathbf{y}_1$. A similar procedure for Φ_2 requires $s_2 \simeq s_4$ and $l_2 \simeq l_4$, and thus $\mathbf{x}_2 \simeq \mathbf{y}_2$. These contractions lead to an exact cancellation $\Phi_2 = 0$, and one gets

$$\begin{aligned}\mathcal{M}_{\rm B}(t) \;&=\; (4\pi\sigma^2)^{\frac{2d_1+d_2}{2}} \int \prod_{i=1}^{2} \mathrm{d}\mathbf{x}_i \mathrm{d}\mathbf{y}_j \mathrm{d}\mathbf{z}_2\; \delta_\sigma(\mathbf{x}_2-\mathbf{y}_2)\\ &\quad\times \sum\; |\mathcal{A}_{s_1}|^2|\mathcal{A}_{s_2}|^2|\mathcal{A}_{s_3}|^2|C_{l_2}| e^{i\left[\delta S_{s_1}-\delta S_{s_3}+\Phi_{12}.\right]}. \qquad (10)\end{aligned}$$

Here, $\delta_\sigma(\mathbf{x}_2-\mathbf{y}_2)$ restricts the spatial integrations to $|\mathbf{x}_2-\mathbf{y}_2| \le \sigma$ because of the finite resolution with which two paths can be equated.

The semiclassical Boltzmann Echo (10) is dominated by two contributions. The first contribution is non diagonal in that all paths are uncorrelated. Applying the central limit theorem one has $\langle \exp\left[i\left\{\delta S_{s_1} - \delta S_{s_3} + \Phi_{12}\right\}\right]\rangle = \exp\left[-\left\langle\delta S^2_{s_1}\right\rangle - \left\langle(\mathcal{S}^{\rm (f)}_{s_1,s_2})^2\right\rangle - \left\langle(\mathcal{S}^{\rm (b)}_{s_1,s_2})^2\right\rangle\right]$, where $\langle\delta S^2_{s_1}\rangle = \int_0^t \mathrm{d}\tau\mathrm{d}\tau'\langle\Sigma_1[\mathbf{q}_{s_1}(\tau)]\;\Sigma_1[\mathbf{q}_{s_1}(\tau')]\rangle$ and $\left\langle(\mathcal{S}^{\rm (f,b)}_{s_1,s_2})^2\right\rangle = \int_0^t \mathrm{d}\tau\;\; \mathrm{d}\tau'\langle\mathcal{U}_{\rm f,b}[\mathbf{q}_{s_1}(\tau),\mathbf{q}_{s_2}(\tau)]\;\; \mathcal{U}_{\rm f,b}[\mathbf{q}_{s_1}(\tau'),\mathbf{q}_{s_2}(\tau')]\rangle$. In chaotic systems, correlators typically decay exponentially fast, thus $\left\langle\delta S^2_{s_1}\right\rangle \simeq \Gamma_{\Sigma_1}\; t$ and $\left\langle(\mathcal{S}^{\rm (f,b)}_{s_1,s_2})^2\right\rangle \simeq \Gamma_{\rm f,b}\; t$.

Finally using the two sum rules

$$(4\pi\sigma^2)^{\frac{d_i}{2}} \int \mathrm{d}\mathbf{x}_i \sum_{s_i} |\mathcal{A}_{s_i}|^2 = 1, \tag{11a}$$

$$\int \mathrm{d}\mathbf{x}_i \int \mathrm{d}\mathbf{y}_i\; \delta_\sigma(\mathbf{y}_i-\mathbf{x}_i)\sum_{l_i}|C_{l_i}| = 1, \tag{11b}$$

one obtains the nondiagonal contribution

$$\mathcal{M}^{\rm (nd)}_{\rm B}(t) \;\simeq\; \exp\left[-\left(\Gamma_{\Sigma_1}+\Gamma_{\rm f}+\Gamma_{\rm b}\;\right)t\right]. \tag{12}$$

The second contribution is diagonal, with $s_1 \simeq s_3$ and $\mathbf{x}_1 \simeq \mathbf{y}_1$. From Eq. (10) it reads

$$\begin{aligned}\mathcal{M}^{\rm (d)}_{\rm B}(t) = (4\pi\sigma^2)^{\frac{2d_1+d_2}{2}} \int \prod_{i=1}^{2} \mathrm{d}\mathbf{x}_i \mathrm{d}\mathbf{y}_i d\mathbf{z}_2\; \delta_\sigma(\mathbf{x}_i-\mathbf{y}_i)\\ \times\sum|\mathcal{A}_{s_1}|^4|\mathcal{A}_{s_2}|^2|C_{l_2}| e^{i\left[\Delta S_{s_1}+\Delta\mathcal{S}^{\rm (f)}_{s_1,s_2}+\Delta\mathcal{S}^{\rm (b)}_{s_1,l_2}\right]}, \qquad (13)\end{aligned}$$

where $\Delta S_{s_1} = \int_0^t \mathrm{d}\tau\nabla_1\Sigma_1[\mathbf{q}_{s_1}(\tau)]\cdot[\mathbf{q}_{s_3}(\tau)-\mathbf{q}_{s_1}(\tau)]$ and $\Delta\mathcal{S}^{\rm (f,b)}_{s_1,s_2} = \int_0^t \mathrm{d}\tau\nabla_1\mathcal{U}_{\rm f,b}[\mathbf{q}_{s_1}(\tau),\mathbf{q}_{s_2}(\tau)]\cdot[\mathbf{q}_{s_3}(\tau)-\mathbf{q}_{s_1}(\tau)]$. We perform a change of coordinates $\int d\mathbf{x_1}\sum|C_{s_1}| = \int d\mathbf{p_1}$, and use both the asymptotics $|C_{s_1}| \propto \exp[-\lambda_1 t]$ valid for chaotic systems [16] and the sum rules of Eqs. (11) to get

$$\mathcal{M}^{\rm (d)}_{\rm B}(t) \;\simeq\; \alpha_1\exp\left[-\lambda_1 t\right]. \tag{14}$$

Here, α_1 is only algebraically time-dependent with $\alpha_1(t=0) = \mathcal{O}(1)$. Together, diagonal (14) and nondiagonal (12) contributions sum up to our main result, Eq. (4). We finally note that the long-time saturation at the inverse Hilbert space size of system 1, $\mathcal{M}_{\rm B}(\infty) = N_1^{-1}$, is obtained from Eq. (8) with the contractions $s_1 \simeq s_3$, $s_2 \simeq s_4$, $l_1 \simeq l_3$ and $l_2 \simeq l_4$.

Analyzing Eq. (4), we first note that $\mathcal{M}_{\rm B}(t)$ depends neither on H_2 nor on Σ_2. This is so because one traces over the uncontrolled degrees of freedom. We stress that this holds even for classically strong Σ_2. Most importantly, besides strong similarities with the Loschmidt Echo, such as competing golden rule and Lyapunov decays [6, 12], the Boltzmann Echo can exhibit a Σ_1-independent decay given by the decoherence rates $\Gamma_{\rm f,b}$

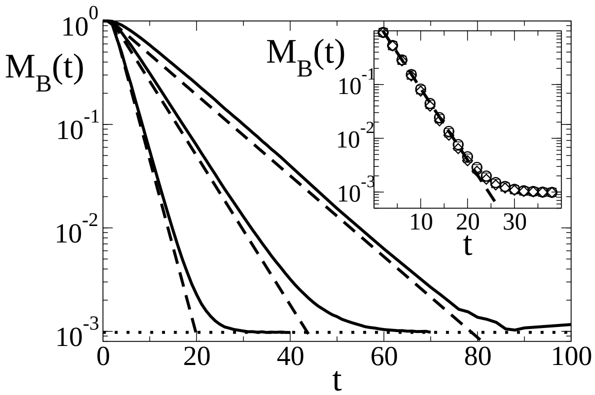

Figure 1: Main plot: Boltzmann echo for $N = 1024$, $K_1 = K_2 = 10.09$, and $\sigma_1 = 0.0018$ ($\Gamma_{\Sigma_1} \simeq 0.09$). Data have been calculated from 50 different initial states. The full lines correspond to $\epsilon = 0$, 0.0018 and 0.0037 (from right to left) and the dashed lines give the predicted exponential decay of Eq. (4), with $\Gamma_{\mathcal{U}} = 1.2\,10^4\epsilon^2, \Gamma_{\Sigma_1} = 2.6\,10^4\sigma_1^2$, $\lambda = 1.6 \gg \Gamma_{\mathcal{U}}, \Gamma_{\Sigma_1}$ (dashed lines have been slightly shifted for clarity). The dotted line gives the saturation N^{-1}. Inset : $\mathcal{M}_{\rm B}$ for $\epsilon = 0.0037$, and $\sigma_1 = 0.0003$ (circles; $\Gamma_{\Sigma_1} \simeq 2.\,10^{-3}$), $\sigma_1 = 0.0006$ (squares; $\Gamma_{\Sigma_1} \simeq 9.\,10^3$), and 0.0009 (diamonds; $\Gamma_{\Sigma_1} \simeq 0.02$). The dashed line indicates the theoretical prediction $\mathcal{M}_{\rm B}(t) = \exp[-0.3t]$.

in the limit $\Gamma_{\Sigma_1} \ll \Gamma_{\rm f,b}$. Extending our analysis to the regime $\Gamma_{\Sigma_1} \ll \delta_1, \Gamma_{\rm f,b} \ll \delta_2$ by means of quantum perturbation theory, we find a gaussian decay of $\mathcal{M}_{\rm B}(t)$, Eq. (5). It is thus possible to reach either a Gaussian or an exponential, Σ_1-independent decay, depending on the balance between the accuracy Σ_1 with which the time-reversal operation is performed and the coupling between controlled and uncontrolled degrees of freedom. This might explain the experimentally observed saturation of the polarization echo as Σ_1 is reduced [15], though a more precise analysis of these experiments in the light of the results presented here is necessary.

We numerically illustrate our findings. We consider two coupled kicked rotators with Hamiltonian

$$H_i = p_i^2/2 + K_i \cos(x_i) \sum_n \delta(t - nT), \qquad (15a)$$

$$\mathcal{U} = \epsilon \sin(x_1 - x_2 - 0.33) \sum_n \delta(t - nT). \quad (15b)$$

We concentrate on the regime $K_i > 7$, for which the dynamics is fully chaotic with Lyapunov exponent $\lambda_i \approx \ln[K_i/2]$. The time-reversed one-particle Hamiltonians are obtained through $K_i \rightarrow K_i + \sigma_i$. We here restrict ourselves to the case $\mathcal{U} = \mathcal{U}_{\rm f} = \mathcal{U}_{\rm b}$. Both rotators are quantized on the torus with discrete momenta $p_n = 2\pi n/N$, $n = 1, 2, ...N$. The one- and two-particle bandwidths and level spacings are given by $B_1 = 2\pi$, $\delta_1 = 2\pi/N$ and $B_2 = 4\pi$, $\delta_2 = 4\pi/N^2$. For more details on the numerical procedure, we refer the reader to Ref. [18].

We first checked that $\mathcal{M}_{\rm B}(t)$ is independent of K_2 (as long as system 2 remains chaotic) and σ_2, and therefore set $K_2 = K_1$, $\sigma_2 = 0$. The main panel in Fig. 1 shows that for $B_1 > \Gamma_{\Sigma_1} > \delta_1$, $B_2 > \Gamma_{\mathcal{U}} > \delta_2$, Eq. (4) is satisfied. Additionally, the inset of Fig. 1 illustrates that when $\Gamma_{\Sigma_1} \ll 2\Gamma_{\mathcal{U}}$, the observed decay is only sensitive to $\mathcal{U}$, and one effectively obtains a Σ_1-independent decay. Further unshown data confirm the existence of the Lyapunov decay [second term in Eq. (4)]. All our numerical results thus confirm the validity of Eq. (4).

In conclusion we propose to analyze echo experiments in the light of the *Boltzmann echo* of Eq. (2) and (3). Our semiclassical and RMT analysis showed that the decay of $\mathcal{M}_{\rm B}(t)$ saturates at a finite value even when the time-reversal operation is performed with infinite accuracy. Further work should attempt to connect these results with echo experiments [8, 9, 10, 11, 15].

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