Chapter 2

Fundamentals of Stability and Stabilizability Theory

2.1 Introduction

In this chapter we shall give a brief account of various discrete-time stability results to be used frequently throughout this book. Stability is a very important property, not only in discrete-time control system design, but also in the analysis and synthesis of nonlinear dynamic systems. In an attempt to make the book self-contained, we collect, in this chapter, the classical Lyapunov theory of stability for discrete-time systems, the center manifold theory for maps and their applications to discrete-time control system design and analysis.

We begin with a section of the description of discrete-time dynamic systems. We will discuss first finite-dimensional state space models and difference equation representations. In section 2.3, several important concepts related to difference equations are introduced. The concepts are a basis for studying the asymptotic behavior of a solution of the difference equation. Section 2.4 introduces Lyapunov stability theory and LaSalle's Invariance Principle for discrete-time autonomous systems. A converse theorem of Lyapunov in discrete-time is also included in this section. In section 2.5, a necessary condition is presented for a discrete-time autonomous system to be asymptotically stable, which indeed is a discrete version of the Brockett's necessary condition [22] [85]. Section 2.6 deals with center manifold theory for maps and its application to the problem of smooth stabilization.

We suppose that the reader is familiar with linear systems theory [28][79], advanced calculus [143], ordinary differential equations [60] and elementary matrix algebra [39]. The content of this chapter is somewhat incomplete in the sense that we shall not provide the detailed proofs of some stability results such as the converse theorem of Lyapunov and the center manifold theory for maps, which are beyond the scope of this book.

The material included in this chapter can be found in the classical textbooks

[24] [90][91][157][115][116] and the series of papers [49][66][80][19].

2.2 Description of Semidynamical Systems–Difference Equations

Because of the application of digital computers in control systems, the discrete-time dynamic systems, or sampled-data systems, are becoming more and more important in control engineering. Such systems, as illustrated in the last chapter, can be described by a set of difference equations, since the system is observed only at discrete-time; e.g., every minute or every second. In this chapter we study the stability concept of a discrete-time nonlinear dynamic system without input, and assume that the system under consideration is described by a difference equation of the form

$$x(k+1) = f(x(k)) \qquad \forall k = 0, 1, 2, \ldots \tag{2.2.1}$$

where the state $x \in \mathbb{R}^n$ and f is a continuous mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$. Obviously, the state at time $k+1$ is completely determined by the state at time k. The solution to the difference equation (2.2.1) with initial value $x(0) = x_0$ is

$$x(k) = f^k(x_0), \tag{2.2.2}$$

where f^k is the kth iterate composition of the function f. That is, $f^0(x) \triangleq I$ the identity function ($f^0(x) = x$) and $f^k(x) \triangleq f(f^{k-1}(x))$ for all $k = 1, 2, \ldots$. Clearly, the solution $f^k(x_0)$ is continuous with respect to the initial value x_0 if f is continuous. Unlike ordinary differential equations, there are no difficulties about the existence and uniqueness of solutions for difference equations. Moreover, we note that the existence and uniqueness are only in the forward direction of time ($k \geq 0$). This observation leads to the following general definition on discrete-time dynamic systems in $\mathbb{R}^n$.

Definition 2.1 *Let Z_+ be the set of all nonnegative integers ($Z_+ \triangleq \{0, 1, 2, \ldots\}$). A discrete-time dynamic system in $\mathbb{R}^n$ is a mapping $T : Z_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfying for all $n,\ k \in Z_+$ and all $x \in \mathbb{R}^n$;*

(i) $T(0, x) = x$

(ii) $T(k, T(n, x)) = T(k+n, x)$

(iii) T is continuous.

An intrinsic property in the definition is condition (ii), which is the so-called semigroup property and expresses the uniqueness of the solution in the forward direction of time. The mapping T is often called a “semiflow” or a “semidynamical” system, and the term "dynamical system" is used when Z_+ can be replaced by Z, where Z denotes the set of all integers, i.e. $Z = \{0, \pm 1, \pm 2, \ldots\}$. In this case, we see that the mapping T indeed has an inverse [90][77]. Comparing continuous-time nonlinear systems with their discrete-time counterparts, we see that a substantial difference between continuous-time and discrete-time systems is that the semigroups tend to appear where groups will appear in the continuous-time case. For this reason, it is not difficult for the reader to imagine that some of the results proposed in the book are parallel to analogous ones in continuous-time nonlinear systems, but in most of the respects the theory developed here is substantially different from continuous-time nonlinear systems and many new phenomena appear.

By Definition 2.1, every difference equation (2.2.1) defines a discrete-time dynamic system $T : T(k, x_0) = f^k(x_0)$. Conversely, every discrete-time dynamic system has associated with it the difference equation $x(k+1) = f(x(k))$, where $f(x) = T(1, x)$. Because of this reason, we shall focus our attention on the difference equation of the form (2.2.1) throughout this chapter.

2.3 ω-Limit Sets of Motions and Basic Properties

In this section, we introduce the notion, following Birkhoff [7] and LaSalle [90] [91], of the ω-limit set $\Omega(x)$ of the motion $f^k(x)$ associated with the discrete-time system (2.2.1). Then we investigate basic properties of the ω-limit sets. There are many reasons for being interested in the concept of ω-limit sets. First of all, the concept of ω-limit sets is instrumental in the development of LaSalle's Invariance Principle. Secondly, the concept plays a crucial role in studying asymptotic stability of motions. Finally, the ω-limit sets are closely related to the stability theory of Lyapunov. In what follows, we first introduce several concepts related to discrete-time systems of the form (2.2.1).

Definition 2.2 *A state $\bar{x} \in \mathbb{R}^n$ is said to be an equilibrium of the discrete-time system (2.2.1) if*

$$f(\bar{x}) = \bar{x}.$$

From the expression of (2.2.2), it follows that the solution of the difference equation (2.2.1) satisfies

$$x(k) = \bar{x} \quad \forall k \in Z_+.$$

That is, the solution of (2.2.1) which starts at the point $\bar{x}$ stays at $\bar{x}$ forever. Therefore, a solution which passes through $\bar{x}$ at some time remains there for all time. This solution is called the equilibrium solution.

In contrast to an ordinary differential equation, a solution of the difference equation (2.2.1) may reach an equilibrium in finite time.

Definition 2.3 *A subset* $\Omega \in \mathrm{I\!R}^{\mathrm{n}}$ *is said to be positively invariant (or invariant) under the mapping* f *associated with a discrete-time system (2.2.1), if* $x \in \Omega$, *then* $f(x) \in \Omega$.

Clearly, an equilibrium of the system (2.2.1) is invariant under the mapping f.

Definition 2.4 *(Birkhoff) Consider a discrete-time system (2.2.1). Let* $f^k(x)$ *be a solution of (2.2.1) with* $x(0) = x$. *A point* y *is a positive limit point of* $f^k(x)$ *if there exists a subsequence* k_n *(*$k_n \to \infty$ *as* $n \to \infty$*) such that*

$$\lim_{n\to\infty} f^{k_n}(x) = y.$$

The ω*-limit set* $\Omega(x)$ *(or positive limit set) of the motion* $f^k(x)$ *from* x *is the set of all positive limit points of* $f^k(x)$.

Let S be any set in $\mathrm{I\!R}^{\mathrm{n}}$. The distance of $x \in \mathrm{I\!R}^{\mathrm{n}}$ from S is defined by

$$d(x, S) = \inf_{y\in S}\{\|x - y\|\}$$

Thus, a sequence $f^k(x)$ in $\mathrm{I\!R}^{\mathrm{n}}$ is said to approach $S \subset \mathrm{I\!R}^{\mathrm{n}}$ if

$$d(f^k(x), S) = \inf_{y\in S}\{\|f^k(x) - y\|\} \to 0 \text{ as } k \to \infty.$$

Lemma 2.5 *Every* ω*-limit set* $\Omega(x)$ *of the motion* $f^k(x)$ *associated with a discrete-time system (2.2.1) is closed and invariant.*

Proof. To show that $\Omega(x)$ is closed, let $w \triangleq \Omega(x)$ and take a sequence of points $\{y_i\}$ in ω which converges to a limit point y as $i \to \infty$. Therefore, for any given $\varepsilon > 0$, there exists a positive integer N_1 such that

$$\|y_i - y\| < \frac{\varepsilon}{2}, \quad \text{whenever } i \geq N_1.$$

Since $y_i (i \geq N)$ is in ω, by Definition 2.4, there exists a subsequence $\{k_n\}$ such that $k_n \to \infty$ as $n \to \infty$, and

$$\|f^{k_n}(x) - y_i\| \leq \frac{\varepsilon}{2} \quad \text{whenever } n \geq N_2$$

Hence

$$\|f^{k_n}(x) - y\| < \varepsilon \text{ if } n \geq N \triangleq \max\{N_1, N_2\}$$

This, in turn, implies that y is also in ω and therefore the set ω is closed.

To show invariance, suppose now $y \in \omega$. By Definition 2.4, there is a sequence of integers k_n such that $k_n \to \infty$ as $n \to \infty$, and

$$f^{k_n}(x) \to y \in \omega \quad \textit{whenever } n \to \infty.$$

By continuity of the mapping f,

$$f(f^{k_n}(x)) = f^{k_n+1}(x) \rightarrow f(y) \in \omega$$

Hence, $\omega = \Omega(x)$ is positively invariant by Definition 2.3. ■

If the motion $f^k(x)$ of a discrete-time system (2.2.1) is bounded for all $k \in Z_+$, the following result concerning the asymptotic behavior of bounded motions can be proven.

Lemma 2.6 *Consider a discrete-time system (2.2.1) and a particular solution* $x(k) = f^k(x)$ *with* $x(0) = x$*. If* $f^k(x)$ *is bounded for all* $k \in Z_+$*, then the* ω*-limit set* $\Omega(x)$ *is a nonempty, invariant and compact set. Moreover,* $\Omega(x)$ *is the smallest closed set that* $f^k(x)$ *approaches as* $k \rightarrow \infty$*.*

Proof. By Lemma 2.5, it is clear that $\Omega(x)$ is invariant and closed. Since $x(k) = f^k(x)$ is bounded, there is a positive real number M such that

$$\|x(k)\| \leq M \quad \forall k.$$

This implies that $\omega = \Omega(x)$ is bounded because

$$\|y\| \leq M \quad \forall y \in \Omega(x)$$

Hence, by the Heine-Borel theorem [143], a closed and bounded set $\Omega(x)$ must be compact. From the Bolzano-Weierstrass theorem [143] and boundedness of $x(k)$, it follows that there exists at least one limit point in $\Omega(x)$ and so ω is nonempty.

We shall show next that $f^k(x)$ converges to $\Omega(x)$ when $x(k) = f^k(x)$ is bounded for all $k \in Z_+$. Note that $d(f^k(x), \Omega(x))$ is bounded since both $x(k) = f^k(x)$ and $\omega = \Omega(x)$ are bounded. Assume $d(f^k(x), \Omega(x))$ does not approach zero as $k \rightarrow \infty$. Then there must exist a subsequence k_n such that $f^{k_n}(x) \rightarrow y \in \Omega(x)$, and $d(f^{k_n}(x), \Omega(x)) \rightarrow C > 0$ as $n \rightarrow \infty$. But

$$d(f^{k_n}(x), \Omega(x)) \leq d(f^{k_n}(x), y) + d(y, \Omega(x)) = d(f^{k_n}(x), y)$$

Let $n \rightarrow \infty$. We have

$$0 < C \leq \lim_{n\rightarrow\infty} d(f^{k_n}(x), y) = 0,$$

which is a contradiction. This proves that $f^k(x)$ converges to $\Omega(x)$ as $k \rightarrow \infty$.

Suppose $f^k(x)$ converges to any closed set F as $k \rightarrow \infty$. Since $f^k(x) \rightarrow \Omega(x)$ as $k \rightarrow \infty$, clearly $\Omega(x) \subset F$. Hence $\Omega(x)$ is the smallest closed set that $f^k(x)$ approaches as $k \rightarrow \infty$. ■

2.4 Lyapunov's Direct Method, LaSalle's Invariance Principle and Converse Theorem of Lyapunov

In this section, we shall give a general survey of the most important methods and results in discrete-time stability theory. The purpose of this section is to introduce the direct method of Lyapunov and LaSalle's Invariance Principle that will be extensively used throughout this book, in order to discuss stability and stabilizability issues for discrete-time semidynamical systems described by difference equations. Although many readers have an intuitive feeling as to what stability and asymptotic stability mean, the concepts of stability and asymptotic stability are very subtle and rigorously mathematical definitions are necessary. Therefore, in what follows, we first introduce several definitions. Consider a difference equation of the form (2.2.1), i.e.

$$x(k+1) = f(x(k)) \text{ with } x(0) = x_0$$

where as assumed in section 2.2, the mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous. Suppose $\bar{x}$ is an equilibrium for (2.2.1) so that

$$f(\bar{x}) = \bar{x}.$$

Definition 2.7 *The equilibrium $\bar{x}$ is said to be stable if for any given $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$\|x - \bar{x}\| < \delta$$

implies

$$\|f^k(x) - \bar{x}\| < \varepsilon \quad \forall k \geq 0.$$

If the equilibrium $\bar{x}$ is not stable then it is called unstable.

Definition 2.8 *The equilibrium $\bar{x}$ is called convergent if for any given $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$\|f^k(x_0) - \bar{x}\| < \varepsilon \text{ whenever } \|x_0 - \bar{x}\| < \delta.$$

That is, there is a $\delta > 0$ such that $\forall x_0 \in B_\delta(\bar{x}) \triangleq \{x \in \mathbb{R}^n : \ \|x - \bar{x}\| < \delta\}$, $\lim_{k\to\infty} f^k(x_0) = \bar{x}$.

Definition 2.9 *The equilibrium $\bar{x}$ is said to be asymptotically stable if it is stable and convergent.*

Roughly speaking, the concept of stability says that if a system starts from a neighborhood of the equilibrium, then all subsequent motions stay in a corresponding neighborhood of the equilibrium for all future time. Asymptotic stability is stronger. In fact, it requires in addition that all subsequent motions eventually return to the equilibrium.

Let $V(x)$ be a scalar function of x, locally defined in a neighborhood $N_{\bar{x}}$ of the equilibrium $\bar{x} \in \mathrm{I\!R}^{\mathrm{n}}$. $V(\bar{x})$ is said to be positive definite with respect to $\bar{x}$ if

(i) $V(\bar{x}) = 0$.

(ii) $V(x) > 0 \ \forall x \in N_{\bar{x}} - \{\bar{x}\}$.

$V(\bar{x})$ is said to be globally positive definite with respect to $\bar{x}$ if the conditions (i) and (ii) hold for $N_{\bar{x}} = \mathrm{I\!R}^{\mathrm{n}}$.

Lemma 2.10 *Suppose* $V : \mathrm{I\!R}^{\mathrm{n}} \to \mathrm{I\!R}$*, is continuous* $\forall x \in \bar{B}_h(\bar{x}) \overset{\Delta}{=} \{x : \|x - \bar{x}\| \leq h\}$*, with* $V(\bar{x}) = 0$*. For any* $\beta > 0$*, there is* $\alpha > 0$ *such that*

$$\|x - \bar{x}\| \geq \alpha \text{ whenever } x \in \{x : V(x) \geq \beta\}.$$

Proof. Since $V(\bar{x}) = 0$ and $V(x)$ is continuous on $\bar{B}_h(\bar{x})$, we have for any $\beta > 0$, there is $\alpha > 0$ so that

$$|V(x)| < \beta \text{ whenever } \|x - \bar{x}\| < \alpha$$

Hence

$$V(x) \geq \beta \Rightarrow \|x - \bar{x}\| \geq \alpha.$$

■

Lemma 2.11 *Suppose* $V(x)$ *is a continuous and positive definite function with respect to* $\bar{x}$ *on* $\bar{B}_h(\bar{x})$*. For any* ε *satisfying* $0 < \varepsilon \leq h$*, the minimum of* $V(x)$ *on the compact set* $F \overset{\Delta}{=} \{x : \varepsilon \leq \|x - \bar{x}\| \leq h\}$ *is positive. That is,*

$$m \overset{\Delta}{=} \min_{\forall x \in F} \ V(x) > 0.$$

Proof. By assumption, $V(\bar{x}) = 0$ and $V(x) > 0 \ \forall x \in B_h(\bar{x}) - \{\bar{x}\}$. Hence $V(x) > 0 \ \forall x \in F$. Since we are taking the minimum of a continuous function over a compact set F in $\mathrm{I\!R}^{\mathrm{n}}$, m must be positive. ■

We are now in the position to state the basic stability theorem of Lyapunov's direct method.

Theorem 2.12 *(Lyapunov's Stability Theorem) Consider a discrete-time system (2.2.1). Suppose there is a* C^0 *(continuous) positive definite function* $V : \mathbb{R}^{\mathrm{n}} \to \mathbb{R}$, *with* $V(\bar{x}) = 0$, *locally defined on a neighborhood* U *of the equilibrium* $\bar{x} \in \mathbb{R}^{\mathrm{n}}$, *such that*

$$V(f(x)) - V(x) \leq 0 \;\forall x \in U.$$

Then the equilibrium $\bar{x}$ *is stable.*

Proof. By assumption, there exists a real number $h > 0$ so that $\forall x \in \bar{B}_h(\bar{x}) \triangleq \{x : \|x - \bar{x}\| \leq h\}$

$$V(\bar{x}) = 0,\; V(x) > 0 \text{ if } x \neq \bar{x}$$

and

$$V(f(x)) - V(x) \leq 0. \tag{2.4.1}$$

Let ε be a given real number satisfying $0 < \varepsilon \leq h$, and F denotes the set

$$F = \{x : \varepsilon \leq \|x - \bar{x}\| \leq h\}$$

Then the quantity

$$m \triangleq \min_{\forall x \in F} V(x)$$

is positive by Lemma 2.11. By continuity of V, there is a $\delta > 0 (\delta < \varepsilon \leq h)$ such that

$$0 \leq V(x) < m \quad \forall x \in B_\delta(\bar{x}) \triangleq \{x : \|x - \bar{x}\| < \delta\} \tag{2.4.2}$$

We show now that $\forall x(0) = x_0 \in B_\delta(\bar{x})$, the corresponding solution of the difference equation (2.2.1) $x(k) = f^k(x_0)$ must satisfy

$$\|f^k(x_0) - \bar{x}\| < \varepsilon \quad \forall k \in Z_+ \tag{2.4.3}$$

This claim can be proven by induction. Taking $k = 0$, one has

$$\|f^0(x_0) - \bar{x}\| = \|x_0 - \bar{x}\| < \delta < \varepsilon$$

Thus (2.4.3) holds for $k = 0$. Moreover note that $x_0 \in \bar{B}_\delta(\bar{x}) \subseteq B_h(\bar{x})$, so it follows from (2.4.1) and (2.4.2) that

$$0 \leq V(f(x_0)) \leq V(x_0) < m,$$

which implies by the definition of m that

$$\|f(x_0) - \bar{x}\| < \varepsilon.$$

Hence, the inequality (2.4.3) is also true for $k = 1$. Suppose (2.4.3) is satisfied for a positive integer $k = \ell$, i.e.,

$$\|f^\ell(x_0) - \bar{x}\| < \varepsilon.$$

This in turn implies that $f^{\ell}(x_0) \in \bar{B}_h(\bar{x})$ since by the choice of ε, $\varepsilon \leq h$. Therefore, it follows from (2.4.1) that

$$V(f^{\ell+1}(x_0)) \leq V(f^{\ell}(x_0)) \leq \ldots \leq V(x_0).$$

By (2.4.2)

$$V(f^{\ell+1}(x_0)) \leq V(x_0) < m.$$

We then conclude from the definition of m that

$$\|f^{\ell+1}(x_0) - \bar{x}\| < \varepsilon$$

This shows that (2.4.3) holds for $k = \ell+1$, so (2.4.3) holds $\forall k \in Z_+ \triangleq \{0, 1, 2, \ldots\}$. ■

This is the discrete version of stability theorem due to Lyapunov [90][50][154]. The theorem constitutes Lyapunov's direct method in the sense that it establishes stability without using specific knowledge of the solution of the difference equation (2.2.1).

Theorem 2.13 *(Lyapunov's Asymptotic Stability Theorem) Consider a discrete-time nonlinear system (2.2.1). Suppose* V *is* C^0 *and positive definite on a neighborhood* U *of the equilibrium* $\bar{x}$*. Assume that*

$$V(f(x)) - V(x) < 0 \quad \forall x \in U$$

Then the equilibrium $\bar{x}$ *is asymptotically stable.*

Proof. According to Theorem 2.12, we see that the equilibrium $\bar{x}$ of (2.2.1) is stable in the sense of Definition 2.7. By Definitions 2.8 and 2.9, Theorem 2.13 will be proven if we can show that there exists a $\delta > 0$ such that

$$\lim_{k\to\infty} f^k(x_0) = \bar{x} \qquad \text{if } \|x_0 - \bar{x}\| < \delta. \tag{2.4.4}$$

By Theorem 2.12, for any given $\varepsilon > 0$, there is $\delta > 0$ such that

$$\|f^k(x_0) - \bar{x}\| < \varepsilon \qquad \forall x_0 \in B_\delta(\bar{x}), \tag{2.4.5}$$

and $V(f^k(x))$ is non-increasing with respect to $k \in Z_+$. Therefore, $\lim_{k\to\infty} V(f^k(x))$ exists and $\lim_{k\to\infty} V(f^k(x_0)) = a \geq 0$. If $a > 0$, then

$$V(f^k(x_0)) \geq a > 0, \tag{2.4.6}$$

which implies by Lemma 2.10 that there exists $\alpha > 0$ such that

$$\|f^k(x_0) - \bar{x}\| \geq \alpha. \tag{2.4.7}$$

Since $V(f(x)) - V(x)$ is negative on U, by Lemma 2.11,

$$\beta \triangleq \min_{\forall x \in F_\alpha} \{-(V(f(x)) - V(x))\} > 0 \tag{2.4.8}$$

where

$$F_\alpha \triangleq \{x : \alpha \leq \|x - \bar{x}\| \leq h\} \tag{2.4.9}$$

Thus $\forall x_0 \in B_\delta(\bar{x}) \subseteq \bar{B}_h(\bar{x})$ (see the proof of Theorem 2.12),

$$V(f^{k+1}(x_0)) < V(f^k(x_0)) < \ldots < V(x_0) \leq -\beta < 0.$$

Hence, it follows from (2.4.8) that

$$V(f^k(x_0)) \leq V(x_0) - \beta k. \tag{2.4.10}$$

Taking limit on both sides of the inequality (2.4.10) results in

$$0 \leq a = \lim_{k \to \infty} V(f^k(x_0)) < -\infty$$

which is a contradiction. So $a = 0$. That is,

$$lim_{k \to \infty} V(f^k(x_0)) = 0 \tag{2.4.11}$$

This, in turn, implies (2.4.4) immediately. In fact, suppose (2.4.4) does not hold. Then there must exist $\varepsilon_0 > 0$ and a subsequence $k_1 < k_2 < \ldots < k_n,\ k_n \to +\infty$, such that

$$\|f^{k_n}(x_0) - \bar{x}\| \geq \varepsilon_0 > 0$$

By Lemma 2.11, there is $m_0 > 0$ such that $V(f^{k_n}(x_0)) \geq m_0 > 0$, which contradicts (2.4.11). ■

Theorem 2.14 *(Instability Theorem) Consider a discrete-time system (2.2.1). Suppose V can take positive values on any neighborhood of the equilibrium $\bar{x}$. Suppose $V(f(x)) - V(x)$ is positive definite with respect to $\bar{x}$. Then the equilibrium $\bar{x}$ is unstable.*

Proof. By contradiction. Suppose there is $h > 0$ and $\delta > 0$ such that $\forall x_0 \in B_\delta(\bar{x}) \triangleq \{x : \|x - \bar{x}\| < \delta\}$, $\quad V(x_0) > 0$ if $x_0 \neq \bar{x}$ and

$$\|f^k(x_0) - \bar{x}\| < h. \tag{2.4.12}$$

Since

$$V(f(x_0)) - V(x_0) > 0 \quad \forall x_0 \in B_\delta(\bar{x}) - \{\bar{x}\},$$

then

$$V(f^k(x_0)) > V(x_0) > 0 \quad \forall k = 1, 2, \ldots$$

By Lemma 2.10, there is $\alpha > 0$ satisfying

$$h \geq \|f^k(x_0) - \bar{x}\| \geq \alpha > 0 \qquad \forall k = 1, 2, \ldots.$$

Thus, it follows from Lemma 2.11 that there exists $m > 0$ such that

$$V(f(x_0)) - V(x_0) \geq m > 0.$$

Hence

$$V(f^k(x_0)) \geq V(x_0) + mk \quad \forall k = 0, 1, 2, \ldots$$

Let k go to ∞, the above inequality contradicts (2.4.12). ■

We now introduce a slightly different instability theorem in order to prove an important theorem for the first approximation of the system (2.2.1).

Theorem 2.15 *Suppose V takes positive values on any neighborhood of $\bar{x}$ and satisfies*

$$V(f(x)) - V(x) = \lambda V(x) + W(x), \; W(x) \geq 0, \lambda > 1 \tag{2.4.13}$$

on a neighborhood of $\bar{x}$. Then the equilibrium $\bar{x}$ is unstable.

Proof. Assume that the equilibrium $\bar{x}$ is stable. That is, there is $\delta > 0$ and $h > 0$ such that

$$\|f^k(x_0) - \bar{x}\| < h \text{ whenever } x \in B_\delta(\bar{x}).$$

Let $V(x_0) > 0$ for some $x_0 \in B_\delta(\bar{x})$. It follows from (2.4.13) that

$$V(f(x_0)) - V(x_0) \geq \lambda V(x_0) > 0$$

By induction,

$$V(f^k(x_0)) \geq \lambda^k V(x_0) > 0. \tag{2.4.14}$$

Taking limit on both sides of the inequality (2.4.14), we get a contradiction. ■

From Theorems 2.13 and 2.15, it is possible to deduce the following stability result by the linear approximation.

Theorem 2.16 *(Stability in the First Approximation) Consider a discrete-time system (2.2.1) and suppose $\bar{x} = 0$ is an equilibrium of (2.2.1). Let*

$$f(x) = Ax + O(\|x\|) \text{ with } A = \frac{\partial f}{\partial x}(0)$$

If the linear system

$$x_{k+1} = Ax_k \tag{2.4.15}$$

is asymptotically stable (i.e., all the eigenvalues of A are located on the open unit circle), then the origin is an asymptotically stable equilibrium of (2.2.1). If there is an eigenvalue of A located outside of the closed unit circle, then the origin is an unstable equilibrium of (2.2.1).

Proof. Assume that A is asymptotically stable. Then there exists a positive definite matrix P satisfying the Lyapunov equation

$$A^T PA - P = -I \tag{2.4.16}$$

Construct a Lyapunov function $V(x) = x^T Px$, which is positive definite, for the system (2.2.1). It follows from (2.4.16) that

$$\begin{aligned} V(f(x)) - V(x) &= (Ax + O\|x\|)^T P(Ax + O\|x\|) - x^T Px \\ &= -x^T x + 2(Ax)^T P(O\|x\|) + (O(\|x\|))^T P(O(\|x\|)) \end{aligned}$$

Obviously, for any $0 < \varepsilon < 1$ we can choose δ so sufficiently small that

$$V(f(x)) - V(x) \leq -\varepsilon x^T x \quad \forall x \in B_\delta(x) \triangleq \{x : \|x\| < \delta\}$$

By Theorem 2.13, the original is an asymptotically stable equilibrium of (2.4.16).

Now suppose A has an eigenvalue which lies outside of the closed unit circle. Choose sufficiently small $\beta > 0$ so that no eigenvalue of the matrix $A_\beta = \frac{1}{\sqrt{1+\beta}} A$ coincides with the eigenvalues of A and A also has an eigenvalue which is greater than $\sqrt{1+\beta}$. Then by Corollary 6.4 and Proposition 6.5 proposed in [91], there is a matrix P that is either negative definite or indefinite satisfying

$$A_\beta^T PA_\beta - P = -I,$$

or equivalently,

$$A^T PA - P = \beta P - (1 + \beta)I \tag{2.4.17}$$

Let $V(x) = x^T Px$. $V(x)$ takes positive values for some x. Then again for any $0 < \varepsilon < 1$ and δ sufficiently small, we can show that

$$V(f(x)) - V(x) = \beta V(x) + W(x)$$

where $W(x) \geq \varepsilon x^T x$ $\forall x \in B_\delta(x)$. By Theorem 2.15, we conclude that the origin of (2.2.1) is unstable. ■

Lyapunov's asymptotic stability (Theorem 2.13) requires to find a Lyapunov function V which satisfies the strict inequality

$$V(f(x)) - V(x) < 0.$$

In practice, it is often much easier to find Lyapunov functions satisfying $V(f(x)) - V(x) \leq 0$. If this is the case, Lyapunov's stability theory cannot provide any answer to the problem of asymptotic stability for discrete-time nonlinear systems of the form (2.2.1). This was not studied in Lyapunov's original paper, but was developed by LaSalle in 1960–1962, from the concepts and properties of the ω-limit sets.

Theorem 2.17 *(LaSalle's Invariance Principle) Consider a discrete-time nonlinear system (2.2.1). Suppose* V *is* C^0 *and positive definite on a neighborhood* U *of the equilibrium* $\bar{x}$*. Assume that*

$$V(f(x)) - V(x) \leq 0 \quad \forall x \in U.$$

Moreover, suppose the solution $f^k(x_0)$ *of (2.2.1) is in* U *and bounded. Then there is a real number* c *such that any bounded trajectory* $f^k(x_0)$ *approaches to the set* $M \cap V^{-1}(c)$*, where* M *is the largest invariant set contained in the set* $I \triangleq \{x \in \mathrm{I\!R^n} : \mathrm{V(f(x)) = V(x)}\} \cap \mathrm{\bar{U}}$*,* $\bar{U}$ *denotes the closure of* U*.*

Proof. Since $x(k) = f^k(x_0)$ is bounded in U, we see from Lemma 2.6 that the ω-limit set $\omega = \omega(x_0)$ is nonempty, $\omega \in \bar{U}$ and $f^k(x_0)$ tends to $\omega(x_0)$. By assumption, $V(f^k(x_0)) = V(x(k))$ is non-increasing and bounded below, so $\lim_{k\to\infty} V(f^k(x_0)) = c$. For any $y \in \omega(x_0)$ there exists a subsequence such that $f^{k_n}(x_0) \to y$ (by Lemma 2.5), so $V(f^{k_n}(x_0)) \to V(y)$ by continuity of V. Hence $V(y) = c$. This shows that V is constant on the ω-limit set, i.e.,

$$V(\omega) = c \text{ or } \omega \subset V^{-1}(c).$$

Since $\omega(x_0)$ is invariant (Lemma 2.6), so $V(f(\omega)) = V(\omega) = c$. Therefore

$$f^k(x_0) \to \omega \subset \{x \in \mathrm{I\!R^n} : \mathrm{V(f(x)) = V(x)}\} \cap \mathrm{\bar{U}} \cap \mathrm{V^{-1}(c)}$$

By invariance of the ω-limit set, we have $\omega \in M$ and therefore,

$$f^k(x_0) \to M \cap V^{-1}(c).$$

■

As a nice application of LaSalle's Invariance Principle, Lyapunov's Asymptotic Stability Theorem (Theorem 2.13) and the Instability Theorem (Theorem 2.14) can also be derived directly from Theorem 2.17. For the reasons of space, the detailed derivations are omitted. The reader is referred to Corollary 2.7 and Proposition 2.8 of LaSalle [91] for additional details.

We conclude this section by introducing a converse theorem of Lyapunov for difference equations due to Halanay [49] in 1963.

Theorem 2.18 *(Converse Theorem of Lyapunov) Consider a discrete-time nonlinear system (2.2.1). Suppose* f *is Lipschitz continuous near the origin, i.e.*

$$\|f(x) - f(y)\| \leq L_1 \|x - y\| \; \forall \; \|x\| < r_0 \text{ and } \|y\| < r_0$$

Assume that the origin is an asymptotically stable equilibrium of (2.2.1). Then there exists a positive definite function V*, which is Lipschitz continuous near the origin with* $V(f(x)) - V(x)$ *negative, i.e., for some* $r > 0$ *and* $L_2 > 0$*, the following conditions are satisfied.*

(i) $|V(x) - V(y)| \leq L_2\|x - y\| \quad \forall \|x\| < r$ and $\|y\| < r$

(ii) $V(0) = 0$ and $V(x) > 0 \quad \forall x \in B_r(0) - \{0\} \triangleq \{x : 0 < \|x\| < r\}$

(iii) $V(f(x)) - V(x) < 0 \quad \forall x \in B_r(0) - \{0\}$.

2.5 Necessary Conditions for Stability and Stabilizability

In this section, we consider a general discrete-time nonlinear system of the form

$$x_{k+1} = f(x_k, u_k) \tag{2.5.1}$$

where $x_k \in \mathrm{I\!R}^{\mathrm{n}}$ and $u_k \in \mathrm{I\!R}^{\mathrm{m}}$, the map of $f : \mathrm{I\!R}^{\mathrm{n}} \times \mathrm{I\!R}^{\mathrm{m}} \to \mathrm{I\!R}^{\mathrm{n}}$ is smooth (i.e., of class C^∞). We assume $f(0,0) = 0$.

The main goal of this section is to study the question of determining when there exists a smooth state feedback control law $u_k = u(x_k)$ such that the equilibrium $x = 0$ of the closed-loop system is asymptotically stable. First of all, we study necessary conditions for the system (2.5.1) to be asymptotically stabilizable via smooth state feedback. Consider the autonomous difference equation

$$x_{k+1} = F(x_k), \quad F(0) = 0. \tag{2.5.2}$$

One can prove as in the case of continuous-time nonlinear systems [22] [15] the following necessary condition for the system (2.5.1) to be locally asymptotically stable at the origin.

Theorem 2.19 *(Discrete Version of Brockett's Condition)*

A necessary condition for the origin to be locally asymptotically stable equilibrium for the autonomous system (2.5.2) is that the equation

$$x - F(x) = y \tag{2.5.3}$$

must be solvable for all y sufficiently small.

Proof. If $x_{k+1} = F(x_k)$ has $x = 0$ as an asymptotically stable equilibrium, we know from the converse theorem of Lyapunov for the autonomous system (2.5.2) [49] [90] that there exists a positive definite Lyapunov function V which is C^1 near the origin, with $V(F(x)) - V(x)$ negative definite. This turns out that, for some $r > 0$ and some $L > 0$

$$\left\|\frac{\partial V}{\partial x}\right\| \leq L \quad \text{for} \quad \|x\| < r \tag{2.5.4}$$

On the other hand, similar to the arguments of Wilson [158], we can show that there are level sets $V^{-1}(c)$ which are closed surfaces around the origin and are homotopic to spheres. The compactness of $V^{-1}(c)$ enable us to define a quantity

$$\beta \triangleq \max_{x \in V^{-1}(c)} \{V(F(x)) - V(x)\}$$

Obviously, $\beta < 0$. This, in turn, implies that the exists $\varepsilon > 0$ and $c > 0$, with $V^{-1}(c) \subset \{x : \|x\| < r\}$, such that $\forall x \in V^{-1}(c)$

$$V(F(x)) - V(x) \leq \beta < -\varepsilon$$

This condition, together with (2.5.4), implies that the drift dynamics associated with the difference equation

$$x_{k+1} = F(x_k) + y$$

point inward on $V^{-1}(c)$ if $\|y\| < \varepsilon/2L$. In fact, it is easy to show that

$$\begin{array}{rcl} V(x_{k+1}) - V(x_k) & = & V(F(x_k) + y) - V(x_k) \\ & = & V(F(x_k)) - V(x_k) + \dfrac{\partial V}{\partial \alpha}|_{\alpha=\xi} y \\ & < & -\varepsilon + L\|y\| \\ & < & -\varepsilon/2 < 0 \end{array}$$

As Brockett has done in [22], we construct now a continuous map by evaluating at time $k = 1$ the solution of $x_{k+1} = F(x_k) + y$ with initial condition $x_0 = x$, then we obtain a continuous map f from $S \triangleq \{x : V(x) \leq c\}$ into S. Applying the Lefschetz fixed point Theorem we can show as in [22] that since $V^{-1}(c)$ are homotopic to spheres the sublevel sets are contractible so that we may compute for f the Lefschetz number $\wedge_f$ as

$$\wedge_f = \sum_q (-1)^q \text{ Trace } H^q(f) = 1 \neq 0.$$

Therefore, we conclude that this map has a fixed point which must be an equilibrium point of $x_{k+1} = F(x_k) + y$. This, in turn, implies that the equation (2.5.3) must be solvable for all y sufficiently small. ■

As an immediate consequence, we have the following necessary conditions for the existence of a smooth stabilizing feedback control law.

Theorem 2.20 *Consider a discrete-time nonlinear system governed by*

$$x_{k+1} = f(x_k, u_k) \tag{2.5.5}$$

with $f(0,0) = 0$ and f being smooth in a neighborhood of (0,0). A necessary condition for the existence of a smooth feedback control law $u_k = u(x_k)$ which renders (0,0) locally asymptotically stable is that:

(a) the uncontrollable modes of the linearized system of (2.5.5) have no eigenvalues outside of the closed unit circle.

(b) there exists a neighborhood N of (0,0) such that for each $x \in N$ there is a control u_k which steers the solution of $x_{k+1} = f(x_k, u_k)$ from x at $k = 0$ to the origin at $k = \infty$.

(c) the mapping $\varphi : (x, u) \mapsto x - f(x, u)$ is onto an open neighborhood of the origin, i.e., the equation

$$\varphi(x, u) = x - f(x, u(x)) = y$$

is solvable for all y in the neighborhood of the origin.

Proof. It is clear from the first approximation that if the linearized system of the nonlinear system (2.5.5) has an uncontrollable mode associated with eigenvalue whose absolute value greater than one, then the system (2.5.5) cannot be stabilized by any smooth state feedback. Thus, (a) follows immediately. Condition (b) follows from the fact that the existence of the state x of the dynamic system (2.5.5) which cannot be steered to the origin by any u_k yields instabilizability of the system (2.5.5). (c) can be proven readily by setting $F(x) = f(x, u(x))$ and using Theorem 2.19. ■

Remark 2.21 *It is well known that in the case of continuous-time nonlinear systems controllability does not imply stabilization by smooth state feedback [1] [22]. Thus it is of interest to investigate whether this is true as well for discrete-time nonlinear systems. The following example provides a definite answer to this question. Consider a discrete-time nonlinear system*

$$x_{k+1} = 2x_k + u_k^3$$

Note that the linearized systems of the nonlinear system has an uncontrollable mode associated with eigenvalue outside the closed unit circle, thus the system can not be stabilized by any smooth state feedback. The system, however, is clearly globally controllable if there are no bounds on u. More interesting, it is easy to see that the system is globally asymptotically stabilized by the almost C^∞ state feedback law (i.e., $u(x)$ is continuous at $x = 0$ and C^∞ at $\mathbb{R}^n - \{0\}$)

$$u_k = -\left(\frac{3}{2}x_k\right)^{\frac{1}{3}}$$

Remark 2.22 *From Theorem 2.20, we conclude that there is no asymptotically stabilizing smooth state feedback control law for the system*

$$x_{k+1} = x_k z_k$$

$$y_{k+1} = y_k z_k + u_k$$

$$z_{k+1} = z_k + (1 - z_k)(x_k^2 + y_k^2),$$

because choosing $\varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ 0 \end{bmatrix}$ *yields that the equation (2.5.3), which in the present case is*

$$\begin{bmatrix} x - xz \\ \\ y - yz - u(x,y,z) \\ \\ (z-1)(x^2+y^2) \end{bmatrix} = \begin{bmatrix} \varepsilon_1 \\ \\ \varepsilon_2 \\ \\ 0 \end{bmatrix},$$

has no solution for all $0 < |\varepsilon_i| << \infty$, $i = 1, 2$.

Similarly, it can be shown that the system

$$\begin{aligned} x_{k+1} &= y_k + x_k z_k - z_k y_k + u_k \\ y_{k+1} &= x_k + y_k^2 - x_k y_k \\ z_{k+1} &= z_k + (x_k - y_k)(1 + \sin^2 z_k) \end{aligned}$$

is not asymptotically stabilized by any smooth state feedback. However, if the system allows two control inputs, i.e.,

$$\begin{aligned} x_{k+1} &= y_k + x_k z_k - z_k y_k + u_k \\ y_{k+1} &= x_k + y_k^2 - x_k y_k + v_k \\ z_{k+1} &= z_k + (x_k - y_k)(1 + \sin^2 z_k) \end{aligned}$$

then the latter one can be rendered locally asymptotically stable via smooth state feedback. The conclusion follows from the stability in the first approximation.

Inspired by the latter example, we now investigate the local asymptotic stabilizability of a class of discrete-time nonlinear system in the form

$$\left.\begin{aligned} x_{k+1}^1 &= f_1(x_k^1, x_k^2) x_k^1 + B u_k, \ x^1 \in \mathrm{I\!R}^{\mathrm{n_1}}, \ \mathrm{u} \in \mathrm{I\!R}^{\mathrm{m}}, \ \mathrm{B} \in \mathrm{I\!R}^{\mathrm{n_1 \times m}} \\ x_{k+1}^2 &= f_2(x_k^1, x_k^2), \qquad x^2 \in \mathrm{I\!R}^{\mathrm{n_2}} \end{aligned}\right\} \tag{2.5.6}$$

We make the following assumptions:

(H1) the function $f(x) = \begin{bmatrix} f_1(x)\, x^1 \\ f_2(x) \end{bmatrix}$, $x \triangleq \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \in \mathrm{I\!R}^{\mathrm{n}}$, $n = n_1 + n_2$, is C^∞ and has $x = 0$ as an equilibrium.

(H2) $f_2(x^1, x^2) = x^2$ implies $x^1 = 0$

(H3) the Jacobian matrix $\dfrac{\partial f_2}{\partial x_1}(0,0)$ has rank n_2.

We also define

$$\bar{m} = \dim \operatorname{span}\,\{b_1, \ldots, b_m\} = \dim \operatorname{span}\,\{B\}$$

Then the following result can be proved.

Theorem 2.23 *Suppose the system (2.5.6) satisfies (H1) and (H2). Suppose there exists a smooth state feedback law* $u_k = u(x_k^1, x_k^2)$, $u_k \triangleq (u_k^1, \ldots, u_k^m)^T$, *such that the equilibrium* $(x^1, x^2) = (0,0)$ *of the closed-loop system is asymptotically stable, then* $\bar{m} = n_1$.

Proof. Choose $\varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} \in \mathrm{I\!R}^{\mathrm{n_1+n_2}}$, with $\varepsilon_2 = 0$, and examine (2.5.3) which in the present case is of the form

$$\begin{bmatrix} (I - f_1(x^1,\ x^2))\, x^1 - Bu(x^1, x^2) \\ \\ x^2 - f_2(x^1, x^2) \end{bmatrix} = \begin{bmatrix} \varepsilon_1 \\ \\ 0 \end{bmatrix}. \tag{2.5.7}$$

One concludes from (H2) that $x^1 = 0$ and $(f_1(x^1, x^2) - I)\, x^1 = 0$. Thus, solvability of the equation (2.5.7) reduces to solvability of the following equation

$$Bu(0, x^2) \equiv \sum_{i=1}^{m} b_i u^i(0, x^2) = \varepsilon_1, \quad \|\varepsilon_1\| << \infty$$

which can only occur if $\bar{m} = n_1$. ■

Under the additional assumption (H3), we can prove that the converse result of Theorem 2.23 holds as well.

Theorem 2.24 *Suppose the system (2.5.6) satisfies (H1)–(H3). There exists a smooth state feedback law* $u_k = u\,(x_k^1, x_k^2)$, *such that the equilibrium* $(x^1, x^2) = (0,0)$ *of the closed-loop system is asymptotically stable if, and only if,* $\bar{m} = n_1$.

Proof. Necessity is an immediate consequence of Theorem 2.23. To show sufficiency, it is important to observe that under the additional assumption (H3), the linearized system of the discrete-time nonlinear system (2.5.6), i.e.,

$$x_{k+1} = \bar{A}x_k + \bar{B}u_k$$

with

$$\bar{A} = \begin{bmatrix} * & 0 \\ A & * \end{bmatrix}, \bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, A = \frac{\partial f_2}{\partial x_1}(0,0)$$

is controllable. As a matter of fact, a simple computation shows that

$$\text{rank}\ \left[\bar{B}\ \bar{A}\bar{B}\ldots\bar{A}^{n_1+n_2-1}\bar{B}\right] = \text{rank}\ [\bar{B}\quad \bar{A}\bar{B}] = \text{rank}\ \begin{bmatrix} B & 0 \\ 0 & AB \end{bmatrix} = n_1+n_2$$

This, in view of the first approximation theory, completes the proof of sufficiency. ■

Remark 2.25 *By Theorem 2.23, there is no stabilizing smooth state feedback control law for the following system*

$$\begin{bmatrix} x^1_{k+1} \\ \\ x^2_{k+1} \end{bmatrix} = A\left(x^1_k, x^2_k, x^3_k, x^4_k\right) \begin{bmatrix} x^1_k \\ \\ x^2_k \end{bmatrix} + b_1 u^1_k + b_2 u^2_k$$

$$\begin{bmatrix} x^3_{k+1} \\ \\ x^4_{k+1} \end{bmatrix} = \begin{bmatrix} (x^4_k)^2+1 & 0 \\ \\ \star & 1 \end{bmatrix} \begin{bmatrix} x^1_k \\ \\ x^2_k \end{bmatrix} + \begin{bmatrix} x^3_k \\ \\ x^4_k \end{bmatrix}$$

where $A(x) \in \mathrm{I\!R}^{2\times 2}$, $b_i \in \mathrm{I\!R}^2$, $u^i_k \in \mathrm{I\!R}$ *for* $i = 1, 2,$ *and* $\dim\text{span}\ \{b_1, b_2\} = 1$.

If, however, $\dim\text{span}\ \{b_1, b_2\} = \bar{m} = 2 = n_1$, *then we conclude from Theorem 2.24 that the system is locally asymptotically stabilizable via smooth state feedback.*

2.6 Center Manifold Theory for Maps and Applications

It has been shown in Theorem 2.16 that the asymptotic stability of the system (2.2.1) is completely determined by the behavior of the linear approximation system (2.4.15) in the following two cases:

(i) All the eigenvalues of $A = \frac{\partial f}{\partial x}(0)$ are located in the open unit circle.

(ii) There is an eigenvalue of A which lies outside the closed unit disc.

This result is known as the principle of stability by the linear approximation. However, the linear approximation approach is unsatisfactory and restricted in the sense that this approach does not give any answer to the asymptotic stability problem of the system (2.2.1) when some eigenvalues of A are located on the unit circle. The case of a system whose matrix A has some eigenvalue with modulus one is commonly referred to as a critical case. In this section, we review an important result for the asymptotic stability analysis, known as center manifold theory for maps which is very useful in analyzing critical cases. We also briefly discuss some direct application of center manifold theory for maps in designing discrete-time nonlinear systems.

For reasons of space, we will not provide the detailed proofs of center manifold theory for maps; the reader is referred to Carr [24] for additional references.

Consider a nonlinear difference equation of the form

$$\left.\begin{aligned} x_{k+1} &= Ax_k + f(x_k, y_k) \\ y_{k+1} &= By_k + g(x_k, y_k) \end{aligned}\right\} \tag{2.6.1}$$

where $x \in \mathrm{I\!R}^{\mathrm{n}}$, $\mathrm{y} \in \mathrm{I\!R}^{\mathrm{m}}$, A and B are square matrices, with dimensions $n \times n$ and $m \times m$, respectively. Assume that each eigenvalue of A has modulus 1 (i.e., located on the unit circle) and each eigenvalue of B has modulus less than 1 (i.e., all the eigenvalues of B lie on the open unit disc). f and g are C^2, vanishing at $(x, y) = (0, 0)$ together with their first order partial derivatives.

Theorem 2.26 *(Existence of Center Manifold for Maps):*
Consider a map $T : \mathrm{I\!R}^{\mathrm{n+m}} \to \mathrm{I\!R}^{\mathrm{n+m}}$ *having the form*

$$T(x, y) = (Ax + f(x, y), By + g(x, y)) \tag{2.6.2}$$

There exists a center manifold $\pi : \mathrm{I\!R}^{\mathrm{n}} \to \mathrm{I\!R}^{\mathrm{m}}$ *for* T*. More precisely, there is a* C^2 *function* $\pi : \mathrm{I\!R}^{\mathrm{n}} \to \mathrm{I\!R}^{\mathrm{m}}$*, with* $\pi(0) = 0$ *and* $\pi'(0) = 0$*, such that*

$$(x_1, y_1) = T(x, \pi(x)) \Rightarrow y_1 = \pi(x_1) \text{ for } |x| < \varepsilon.$$

The function π *can be determined by solving the functional equation in the form*

$$\pi(Ax + f(x, \pi(x))) = B\pi(x) + g(x, \pi(x)) \tag{2.6.3}$$

Once a solution $y = \pi(x)$ satisfying (2.6.3), with $\pi(0) = 0$ and $\pi'(0) = 0$, is available, it can be applied to analyze the asymptotic stability. The following result shows to what extent center manifolds are useful in the analysis of the asymptotic properties of the system (2.6.1) near $(x, y) = (0, 0)$. Recall that any trajectory of (2.6.1) starting a point $(x_0, y_0) = (x_0, \pi(x_0))$ (which is a point near $(x, y) = (0, 0)$) of a center manifold can be described by

$$y_k = \pi(\xi_k),\ \xi_k = x_k$$

with ξ_k the solution of the difference equation

$$\xi_{k+1} = A\xi_k + f(\xi_k, \pi(\xi_k)),\ \xi_0 = x_0 \tag{2.6.4}$$

Now we are in a position to state the following reduction principle.

Theorem 2.27 *(Reduction Principle):*

(a) *Suppose that the equilibrium* $\xi = 0$ *of (2.6.4) is stable (asymptotically stable) (unstable). Then the equilibrium* $(x, y) = (0, 0)$ *of (2.6.1) is stable (asymptotically stable) (unstable).*

(b) *Suppose the equilibrium* $(x, y) = (0, 0)$ *of (2.6.1) is stable. Let* (x_k, y_k) *be a solution of (2.6.1) with* (x_0, y_0) *sufficiently small. Then there is a solution* ξ_k *of (2.6.4) such that*

$$\|x_k - \xi_k\| \leq Ma^k \text{ and } \|y_k - \pi(\xi_k)\| \leq Ma^k$$

for all k*, where* M *and* a *are positive constants with* $0 < a < 1$.

The importance and advantage of the reduction principle are that the asymptotic behavior of an $(n + m)$-dimensional system (2.6.1), locally speaking, is completely determined by its behavior on the center manifold, namely an n-dimensional subsystem (2.6.4), which has a lower dimension than (2.6.1). Now a natural question arises immediately in practical applications: How can one solve the center manifold π from the functional equation (2.6.3)? This is in general quite difficult. The next result, however, shows that, in principle, it is always possible to approximate the solution $y = \pi(x)$ (center manifold) of the functional equation (2.6.3) to any degree of accuracy and, then, to use the approximate solution in the reduced equation (2.6.4). In this way, we are able to determine the locally asymptotic properties of the equilibrium $\xi = 0$ of (2.6.4) and, thus, the asymptotic stability of the equilibrium $(x, y) = (0, 0)$ of (2.6.1).

Theorem 2.28 *(Approximation Property)*
Let $\phi : \mathrm{I\!R}^{\mathrm{n}} \to \mathrm{I\!R}^{\mathrm{m}}$ *be a* C^1 *map, with* $\phi(0) = 0$ *and* $\phi'(0) = 0$*. Define a map* $M\phi$ *by*

$$(M\phi)(x) = \phi(Ax + f(x, \phi(x))) - (B\phi(x) + g(x, \phi(x)) \tag{2.6.5}$$

Suppose $(M\phi)(x) = 0(|x|^r)$ *as* $x \to 0$ *for some* $r > 1$*. Then for* $x \to 0$,

$$\pi(x) = \phi(x) + 0(|x|^r).$$

To illustrate the application of the center manifold theory for maps, we next study, via Theorems 2.26–2.28, the stabilizability problem of discrete-time nonlinear systems having the normal form

$$\eta_{k+1} = f(\eta_k, \xi_k) \tag{2.6.6}$$

$$\xi_{k+1} = A\xi_k + Bu_k \tag{2.6.7}$$

where $\eta_k \in \mathrm{I\!R}^{\mathrm{n-r}}$, $\mathrm{\xi_k} \in \mathrm{I\!R}^{\mathrm{r}}$ and $u_k \in \mathrm{I\!R}^{\mathrm{m}}$, the map $f : \mathrm{I\!R}^{\mathrm{n}} \to \mathrm{I\!R}^{\mathrm{n-r}}$ is smooth (i.e. of class C^∞). A and B are constant matrices with dimensions $r \times r$ and $r \times m$, respectively.

If the pair (A, B) is controllable and the zero dynamics of (2.6.6)–(2.6.7), namely

$$\eta_{k+1} = f(\eta_k, 0)$$

are locally (resp. globally) asymptotically stable at the equilibrium $\eta = 0$, the discrete-time nonlinear systems (2.6.6)–(2.6.7) is said to be minimum-phase. As a matter of fact, let $y = C\xi$ be the output with the pair (C, A) being observable, then the assumptions above imply that (2.6.6)–(2.6.7) with $y = C\xi$ is a minimum-phase nonlinear system.

Our interest in this class of discrete-time nonlinear systems is inspired by the concepts of normal forms, zero dynamics and minimum-phase concepts introduced for continuous-time nonlinear systems in the recent series of publications [8]–[12], [69]. It has been shown that a continuous-time minimum-phase nonlinear system can always be locally stabilizable by smooth state feedback [9]. Moreover, a minimum-phase nonlinear system having relative degree $\{1, 1, \ldots, 1\}$ is globally asymptotically stabilizable via smooth state feedback [11] [12]. Now we want to investigate to what extent the results proposed in [11] [12] can be extended to discrete-time nonlinear systems of the from (3.3.7)–(3.3.8). It turns out, as expected, that most of the local stabilization results in continuous-time nonlinear systems [9], [11] can be extended to their discrete-time counterparts, by means of the center manifold theory for maps. The following lemma shows how the local stabilization result developed in [9] can be carried out analogously in the discrete-time nonlinear context.

Lemma 2.29 *(Triangular Structure Stability)*

Consider a discrete-time nonlinear system

$$\eta_{k+1} = f(\eta_k, \xi_k), \; f \in C^\infty \tag{2.6.8}$$

$$\xi_{k+1} = A\xi_k + R(\eta_k, \xi_k) \tag{2.6.9}$$

Suppose that $R(\eta, 0) = 0$ *for all* η *near 0 and* $\frac{\partial R}{\partial \xi}(0, 0) = 0$. *Suppose* $\eta = 0$ *is an asymptotically stable equilibrium of* $\eta_{k+1} = f(\eta_k, 0)$ *and all the eigenvalues of* A *are located inside the unit disc. Then* $(\eta, \xi) = (0, 0)$ *is an asymptotically stable equilibrium of the system (2.6.8)–(2.6.9).*

Proof. Rewriting

$$f(\eta_k, \xi_k) = F\eta_k + G\xi_k + g(\eta_k, \xi_k)$$

and using a linear change of coordinates $z_k = (z_k^{1^T}, z_k^{2^T})^T = T\eta_k + K\xi_k$, one changes the system (2.6.8)–(2.6.9) into the system

$$\Sigma \begin{cases} z_{k+1}^1 = F_1 z_k^1 + g_1(z_k^1, z_k^2, \xi_k) \\ z_{k+1}^2 = F_2 z_k^2 + G_2\xi_k + g_2(z_k^1, z_k^2, \xi_k) \\ \xi_{k+1} = A\xi_k + \tilde{R}(z_k^1, z_k^2, \xi_k) \end{cases}$$

where all the eigenvalues of F_1 are located on the unit circle and all the eigenvalues of F_2 lie inside the unit circle. Moreover, the functions g_1 and g_2 vanish at $(z^1, z^2, \xi) = (0, 0, 0)$ together with their first-order partial derivatives. The function $\tilde{R}$ satisfies $\tilde{R}(z^1, z^2, 0) = 0$ for all z near 0 and $\frac{\partial \tilde{R}}{\partial \xi}(0, 0, 0) = 0$.

By assumption, the equilibrium $(z^1, z^2) = (0, 0)$ of the system

$$\begin{aligned} z^1_{k+1} &= F_1 z^1_k + g_1(z^1_k, z^2_k, 0) \\ z^2_{k+1} &= F_2 z^2_k + g_2(z^1_k, z^2_k, 0) \end{aligned}$$

is asymptotically stable. By center manifold theory for maps [24], there exists a center manifold $z^2 = \pi(z^1)$ for the map

$$T : (z^1, z^2) \to (F_1 z^1 + g_1(z^1, z^2, 0), F_2 z^2 + g_2(z^1, z^2, 0))$$

at $(z^1, z^2) = (0, 0)$. Moreover, π is determined by the equation

$$F_2 \pi(z^1) + g_2(z^1, \pi(z^1), 0) = \pi(F_1 z^1 + g_1(z^1, \pi(z^1), 0)) \tag{2.6.10}$$

Suppose now the function π can be solved from (2.6.10). Once π is available, then by the reduction principle [24] the reduced dynamic subsystem

$$z^1_{k+1} = F_1 z^1_k + g_1(z^1_k, \pi(z^1_k), 0)$$

has necessarily $z^1 = 0$ as an asymptotically stable equilibrium.

Now let us consider the full system (2.6.8)–(2.6.9) (or equivalently, the full system Σ). Recall that each eigenvalue of A has modulus less than 1, a center manifold

$$\begin{bmatrix} \xi \\ z^2 \end{bmatrix} = \begin{bmatrix} \pi_1(z^1) \\ \pi_2(z^1) \end{bmatrix}$$

exists for the system Σ and satisfies

$$A\pi_1(z^1) + \tilde{R}(z^1, \pi_2(z^1), \pi_1(z^1)) = \pi_1(F_1 z^1 + g_1(z^1, \pi_2(z^1), \pi_1(z^1)))$$

$$F_2 \pi_2(z^1) + G_2 \pi_1(z^1) + g_2(z^1, \pi_2(z^1), \pi_1(z^1)) = \pi_2(F_1 z^1 + g_1(z^1, \pi_2(z^1), \pi_1(z^1)))$$

From (2.6.10) it is clear that

$$\pi_1(z^1) = 0, \quad \pi_2(z^1) = \pi(z^1)$$

are a set of solutions to the above equation. Using again the reduction principle, we conclude that the full system Σ has $(z^1, z^2, \xi) = (0, 0, 0)$ as an asymptotically stable equilibrium if the reduced dynamic subsystem

$$z_{k+1} = F_1 z^1_k + g_1(z^1_k, \pi(z^1_k), 0)$$

is asymptotically stable at the equilibrium $z^1 = 0$. But this is indeed the case according to the arguments above. ■

As an immediate consequence of Lemma 2.29, we have the following stabilization result.

Theorem 2.30 *Consider a discrete-time nonlinear system of the form (2.6.6)–(2.6.7). Suppose the pair (A, B) is controllable and the zero dynamics of the system, namely $\eta_{k+1} = f(\eta_k, 0)$, has $\eta = 0$ as an asymptotically stable equilibrium. Then the full system (2.6.6)–(2.6.7) is asymptotically stabilizable by smooth state feedback.*

Theorem 2.30 is indeed a discrete analogous result given in [9] and [11]. It is important to observe that, for a special class of discrete-time nonlinear systems (2.6.6)–(2.6.7), we even have the following global stabilization theorem.

Theorem 2.31 *(Relative degree $\{1, \ldots, 1\}$). Consider a special case of Theorem 2.30 where B is a square matrix of dimensional $m \times m$ and rank $B = m$. Suppose $\eta_{k+1} = f(\eta_k, 0)$ is is globally asymptotically stable at the equilibrium $\eta = 0$. Then, the system (2.6.6)–(2.6.7) is globally asymptotically stabilizable by smooth state feedback.*

Theorem 2.31, which is viewed as a discrete-time counterpart of Proposition 6.3 presented in [11], can be deduced directly from the following auxiliary result.

Theorem 2.32 *Consider a discrete-time nonlinear system of the form*

$$
\begin{aligned}
x_{k+1} &= f\left(x_k, y_k\right) && (2.6.11)\\
y_{k+1} &= u_k && (2.6.12)
\end{aligned}
$$

where, $x \in \mathrm{I\!R}^{\mathrm{n}}$, $y \in \mathrm{I\!R}^{\mathrm{m}}$ and $f : \mathrm{I\!R}^{\mathrm{n\times m}} \rightarrow \mathrm{I\!R}^{\mathrm{n}}$, is a smooth mapping. Suppose there exists a smooth feedback

$$y_k = \varphi(x_k) \quad \text{with} \quad \varphi(0) = 0$$

such that $x_{k+1} = f\left(x_k, \varphi(x_k)\right)$ has $x_k = 0$ as a locally (resp. globally) asymptotically stable equilibrium. Then the overall system (2.6.11)–(2.6.12) is locally (resp. globally) asymptotically stabilizable (LAS or GAS for short) by smooth state feedback.

Proof. Choosing a smooth state feedback law $u_k = \varphi\left(f(x_k, y_k)\right)$ results in the closed-loop system

$$x_{k+1} = f(x_k, y_k) \tag{2.6.13}$$

$$y_{k+1} = \varphi\left(f(x_k, y_k)\right) \tag{2.6.14}$$

Define

$$f_\varphi(x_k) \triangleq f\left(x_k, \varphi(x_k)\right)$$

Then it is clear that

$$\begin{aligned}
x_{k+1} &= f(x_k, y_k) \\
&= f\left[f(x_{k-1}, y_{k-1}), \varphi\left(f(x_{k-1}, y_{k-1})\right)\right] \\
&= f_\varphi\left(f(x_{k-1}, y_{k-1})\right) \\
&= f_\varphi\left(f\left[f(x_{k-2}, y_{k-2}), \varphi\left(f(x_{k-2}, y_{k-2})\right)\right]\right) \\
&= f_\varphi(f_\varphi\left(f(x_{k-2}, y_{k-2})\right)) \\
&\triangleq f_\varphi^2\left(f(x_{k-2}, y_{k-2})\right) \\
&= \ldots \\
&= f_\varphi^k\left(f(x_0, y_0)\right).
\end{aligned}$$

Hence, all the trajectories of the closed-loop system (2.6.13)–(2.6.14) are characterized by

$$x_{k+1} = f_\varphi^k\left(f(x_0, y_0)\right)$$

$$y_{k+1} = \varphi\left(f_\varphi^k\left(f(x_0, y_0)\right)\right)$$

By assumption,

$$\lim_{k\to\infty} f_\varphi^k\left(x^0\right) = 0 \quad \forall x^0 \in \mathrm{I\!R}^{\mathrm{n}} \tag{2.6.15}$$

and

$$\lim_{x^0\to 0} f_\varphi^k(x^0) = 0 \quad \forall k \geq 0 \tag{2.6.16}$$

Recall that f is a smooth mapping from $\mathrm{I\!R}^{\mathrm{n}\times\mathrm{m}}$ to $\mathrm{I\!R}^{\mathrm{n}}$. We conclude from (2.6.15) and (2.6.16) that for any $(x_0, y_0) \in \mathrm{I\!R}^{\mathrm{n}} \times \mathrm{I\!R}^{\mathrm{m}}$,

$$\lim_{k\to\infty} \begin{bmatrix} x_{k+1} \\ \\ y_{k+1} \end{bmatrix} = \lim_{k\to\infty} \begin{bmatrix} f_\varphi^k\left(f(x_0, y_0)\right) \\ \\ \varphi\left(f_\varphi^k\left(f(x_0, y_0)\right)\right) \end{bmatrix} = 0$$

and

$$\lim_{(x_0,y_0)\to 0} \begin{bmatrix} x_{k+1} \\ \\ y_{k+1} \end{bmatrix} = \lim_{(x_0,y_0)\to 0} \begin{bmatrix} f_\varphi^k\left(f(x_0, y_0)\right) \\ \\ \varphi\left(f_\varphi^k\left(f(x_0, y_0)\right)\right) \end{bmatrix} = 0.$$

This, in view of the definition of asymptotic stability (in the sense of of Lyapunov), completes the proof. ■

As in the continuous-time case [82], [83] and [10]–[11], we can deduce from Theorem 2.32 a number of corresponding stabilization results for discrete-time nonlinear systems with triangular structure. The first result is the stabilizability for the system with a chain of "integrator".

Corollary 2.33 *If a discrete-time nonlinear system* $x_{k+1} = f(x_k, u_k)$ *is smooth stabilizable, then the cascade system*

$$\left.\begin{aligned} x_{k+1} &= f\left(x_k, \xi_k^1\right) \\ \xi_{k+1}^1 &= \xi_k^2 \\ &\vdots \\ \xi_{k+1}^{r-1} &= \xi_k^r \\ \xi_{k+1}^r &= u_k \end{aligned}\right\} \tag{2.6.17}$$

is smooth stabilizable as well. In particular, if $x_{k+1} = f(x_k, u_k)$ *is LAS (resp. GAS) by* $u_k = \varphi(x_k)$*, then the system (2.6.17) is LAS (resp. GAS) by*

$$u_k = \varphi\left(f\left\{f\left[f\left(\ldots f\left(f\left(x_k, \xi_k^1\right), \xi_k^2\right)\ldots\right), \xi_k^{r-1}\right], \xi_k^r\right\}\right) \tag{2.6.18}$$

Proof. To make the idea clear, we first examine the case where $r = 2$. Let

$$\tilde{x}_k \triangleq \begin{bmatrix} x_k \\ \\ \xi_k^1 \end{bmatrix} \text{ and } \mathrm{F}\left(\mathrm{\tilde{x}_k}, \xi_\mathrm{k}^2\right) \triangleq \begin{bmatrix} f(x_k, \xi_k^1) \\ \\ \xi_k^2 \end{bmatrix} \tag{2.6.19}$$

Then, the system (2.6.17) can be represented as

$$\left.\begin{aligned} \tilde{x}_{k+1} &= F\left(\tilde{x}_k, \xi_k^2\right) \\ \xi_{k+1}^2 &= u_k \end{aligned}\right\} \tag{2.6.20}$$

which has the exactly same form of (2.6.11)–(2.6.12). By assumption and Theorem 2.32, system (2.6.20) is smoothly stabilizable. In particular, (2.6.20) can be stabilized by the smooth state feedback law

$$u_k = \varphi\left(f\left(F(\tilde{x}_k, \xi_k^2)\right)\right),$$

Or, equivalently,

$$u_k = \varphi\left(f\left(f(x_k, \xi_k^1), \xi_k^2\right)\right) \tag{2.6.21}$$

Using an inductive argument, it is straightforward to verify that the cascade system (2.6.17) is smoothly stabilizable and a particular feedback control law is determined by (2.6.18). ■

The second result is the stabilization Theorem for a class of discrete-time nonlinear systems in the special “strict feedback form”. The corresponding theory in continuous-time has been presented in [82].

Corollary 2.34 *Consider a discrete-time nonlinear system in strict feedback form*

$$\left.\begin{aligned} x^1_{k+1} &= f_1(x^1_k, x^2_k) \\ x^2_{k+1} &= x^3_k + f_2(x^1_k, x^2_k) \\ &\vdots \\ x^{r-1}_{k+1} &= x^r_k + f_{r-1}\left(x^1_k, \ldots, x^{r-1}_k\right) \\ x^r_{k+1} &= u_k + f_r\left(x^1_k, \ldots, x^r_k\right) \end{aligned}\right\} \tag{2.6.22}$$

Suppose the subsystem

$$x^1_{k+1} = f_1\left(x^1_k, x^2_k\right)$$

is smoothly stabilizable, then the overall system (2.6.22) is smoothly stabilizable as well.

Proof. It suffices to show that the system (2.6.22) is feedback equivalent to the system (2.6.17)—a chain of integrators form. The proof of Corollary 2.34 can be better understood by studying the case where $r = 3$. In the case when $r = 3$, the system (2.6.22) reduces to

$$\begin{aligned} x^1_{k+1} &= f_1\left(x^1_k, x^2_k\right) \\ x^2_{k+1} &= x^3_k + f_2\left(x^1_k, x^2_k\right) \\ x^3_{k+1} &= u_k + f_3\left(x^1_k, x^2_k, x^3_k\right) \end{aligned} \tag{2.6.23}$$

Define

$$\tilde{x}^3_k \triangleq x^3_k + f_2\left(x^1_k, x^2_k\right)$$

Then

$$\begin{aligned} \tilde{x}^3_{k+1} = {} & x^3_{k+1} + f_2\left(x^1_{k+1}, x^2_{k+1}\right) \\ = {} & u_k + f_3\left(x^1_k, x^2_k, \tilde{x}^3_k - f_2\left(x^1_k, x^2_k\right)\right) + f_2\left(f_1\left(x^1_k, x^2_k\right), \tilde{x}^3_k\right) \end{aligned}$$

Obviously, setting $\tilde{x}^3_{k+1} \triangleq v_k$ yields

$$\begin{aligned} x^1_{k+1} &= f_1\left(x^1_k, x^2_k\right) \\ x^2_{k+1} &= \tilde{x}^3_k \\ \tilde{x}^3_{k+1} &= v_k \end{aligned} \tag{2.6.24}$$

which is feedback equivalent to the system (2.6.23). Since the subsystem $x^1_{k+1} = f_1\left(x^1_k, x^2_k\right)$ is smoothly stabilizable, it follows immediately from Corollary 2.33 that

(2.6.24) is smoothly stabilizable. Therefore, the original system (2.6.23) in "strict feedback form" is also smoothly stabilizable. In particular, if there exists a smooth feedback $x_k^2 = \varphi(x_k^1)$ which stabilizes the system $x_{k+1}^1 = f_1\left(x_k^1, x_k^2\right)$, the system (2.6.24) is smoothly stabilized by (see Corollary 2.33, (2.6.18))

$$\begin{aligned} v_k &= \varphi\left(f_1\left(f_1\left(x_k^1, x_k^2\right), \tilde{x}_k^3\right)\right) \\ &= \varphi\left(f_1\left(f_1\left(x_k^1, x_k^2\right), x_k^3 + f_2\left(x_k^1, x_k^2\right)\right)\right) \end{aligned} \tag{2.6.25}$$

This, in turn, implies that the original system (2.6.23) is stabilized by the smooth state feedback control law

$$\begin{aligned} u_k = \ & \varphi\left\{f_1\left(f_1\left(x_k^1, x_k^2\right), x_k^3 + f_2\left(x_k^1, x_k^2\right)\right)\right\} \\ & -f_2\left(f_1\left(x_k^1, x_k^2\right), x_k^3 + f_2\left(x_k^1, x_k^2\right)\right) - f_3\left(x_k^1, x_k^2, x_k^3\right) \end{aligned}$$

By essentially using the inductive argument, it is not difficult to complete the proof of Corollary 2.34. ■

Remark 2.35 *The Euler (point-slope) discretization of the controlled plant* $\dot{x} = f(x, u)$ *with sampling interval* $h > 0$ *is*

$$x(t+h) = x(t) + hf\left(x(t), u(t)\right)$$

abbreviated as

$$x_{k+1} = x_k + hf\left(x_k, u_k\right) \triangleq \tilde{f}_h\left(x_k, u_k\right)$$

Therefore, it will be worth to investigate sufficient conditions under which the stability results developed so far in discrete-time can be applied to the discretized systems of a class of continuous-time nonlinear systems, such as the normal form [9] [10], the chain of integrators form [82] [11] and the system in strict feedback form [82], which have attracted considerable attention and interest in the geometric nonlinear control literature.