4.2 Center Manifold Theory for Continuous-Time Systems

This section is a basic review of the center manifold theory for continuous-time systems. In the literature, A.L. Kelley proved the existence of a C^{r-1} center manifold at a nonhyperbolic equilibrium point of a finite-dimensional vector field of class C^r (see [Kel67], [Car81]). Recently, A. Vanderbauwhede and S.A. Van Gils have proved the existence of a C^r center manifold at this nonhyperbolic equilibrium point (see [VG87]).

Suppose the matrix A has s eigenvalues with negative real part (σ_s), u eigenvalues with positive real part (σ_u) and c eigenvalues with zero real part (σ_c). Then, it follows from the linear algebra that $\mathbb{R}^n$ can be decomposed into the direct sum of three invariant subspaces, denoted E^s, E^u, E^c (with dimensions s, u, c respectively), with the property that $A|_{E^s}$ has all eigenvalues with negative real part, $A|_{E^u}$ has all eigenvalues with positive real part, and $A|_{E^c}$ has all eigenvalues with zero real part.

We now state the famous *center manifold theorem.* A simple proof for this theorem for the C^0 case is given in Appendix E.2.

Theorem 4.2.1 (**Center Manifold Theorem**) *Let* $\dot{x} = f(x) : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ *be a* $C^r, (r \geq 1)$ *vector field vanishing at the origin,* i.e. $f(0) = 0$ *and* $A = Df(0)$ *whose spectrum* $\sigma(A) = \{\sigma_s, \sigma_u, \sigma_c\}$ *where* $s + u + c = n$*, with their corresponding eigenspaces* E^s, E^u *and* E^c*. Then locally, there exist* C^r *stable, unstable and center manifolds* W^s, W^u *and* W^c *tangent to* E^s, E^u *and* E^c *respectively at* 0 *and invariant for the flow of* f. $\square$

Remark 4.2.1 (*Nonuniqueness of the center manifold*) [Kel67] We consider a simple planar system described by

$$\begin{aligned} \dot{x_1} &= x_1^2, \\ \dot{x_2} &= -x_2. \end{aligned} \tag{4.2.1}$$

Integrating the system (4.2.1) directly, we obtain the general solution

$$\begin{aligned} x_1(t) &= \frac{x_1^0}{1-tx_1^0}, \\ x_2(t) &= x_2^0 e^{-t} \end{aligned}$$

where $x(0) = (x_1^0, x_2^0)$. Eliminating t, we obtain

$$x_2 = (x_2^0 e^{\frac{-1}{x_1^0}}) e^{\frac{1}{x_1}}.$$

The phase portrait in the (x_1, x_2) plane for the system (4.2.1) is illustrated in Figure 4.2.1.

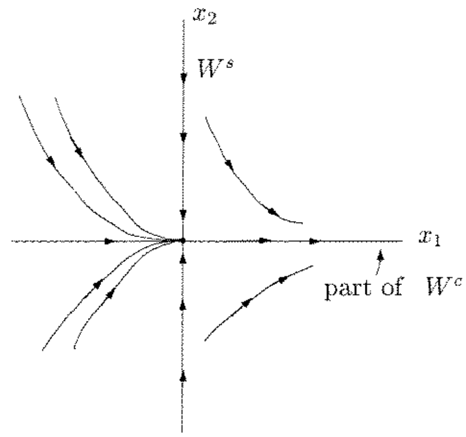

Figure 4.2.1: Ex. 4.1.1: Phase potrait of the plant, showing some center manifolds

Let

$$u(x_1, c) = \begin{cases} c\, e^{\frac{1}{x_1}} & \text{for } y < 0, \\ 0 & \text{for } y \geq 0. \end{cases}$$

Clearly, $u(0, c) = u_{x_1}(0, c) = 0$, so that

$$W^c = \{(x_1, x_2) : x_2 = u(x_1, c),\ c \text{ arbitrary } \}$$

is a center manifold for each real constant c. Thus the center manifold, tangent to the direction of the eigenvector belonging to 0 (the x_1-axis), is far from unique. □

Since we will be mainly interested in the cases in which $x = 0$ can be a stable equilibrium, in what follows, we will restrict ourselves to consider only those cases in which the matrix $A = Df(0)$ has all eigenvalues with nonpositive real part.

When A has all eigenvalues with nonpositive real part, one can always make a linear change of coordinates in U such that the system $\dot{x} = f(x)$ is represented in the form

$$\begin{aligned} \dot{y} &= Sy + \psi(y, z), \\ \dot{z} &= Cz + \mu(y, z), \end{aligned} \tag{4.2.2}$$

where S is an $(s \times s)$ stable (or Hurwitz) matrix (*i.e.* with all eigenvalues having negative real part), C is an $(c \times c)$ "central" matrix (*i.e.* with all eigenvalues lying on the imaginary axis), and the functions ψ and μ are C^r functions vanishing at $(y, z) = (0, 0)$ together with all their first order derivatives.

The center manifold W^c goes through the origin, where it is tangent to E^c, *i.e.* the z-axis where $y = 0$. Hence, it can be represented as

$$W^c = \{(y, z) \in \mathbb{R}^s \times V \mid y = \pi(z),\ \pi(0) = 0 \text{ and } \pi'(0) = 0\},$$

where V is a neighborhood of $z = 0$ in $\mathbb{R}^c$.

Remark 4.2.2 [Kel67] If the origin is Lyapunov stable with respect to a center manifold for the system (4.2.2), then it can be shown that the center manifold is unique. □

Remark 4.2.3 W^c is invariant under the flow of Eq. (4.2.2) and this imposes a constraint on π that can be easily deduced in the following way. Let $(y(t), z(t))$ be any solution curve of Eq. (4.2.2) that lies on the center manifold W^c, *i.e.* suppose that $y(t) = \pi(z(t))$. Differentiating this with respect to t, we obtain the relation

$$S\pi(z(t)) + \psi(\pi(z(t)), z(t)) = \frac{\partial \pi}{\partial z}\,(Cz(t) + \mu(\pi(z(t)), z(t))). \tag{4.2.3}$$

Since the condition (4.2.3) must hold for any solution curve of Eq. (4.2.2), we conclude that the mapping π satisfies the partial differential equation

$$\frac{\partial \pi}{\partial z}\,(Cz + \mu(\pi(z), z)) = S\pi(z) + \psi(\pi(z), z). \tag{4.2.4}$$

Remark 4.2.4 Consider, instead of Eq. (4.2.2), a system of the form

$$\begin{aligned} \dot{y} &= Sy + Tz + \psi(y, z), \\ \dot{z} &= Cz + \mu(y, z), \end{aligned} \tag{4.2.5}$$

where S is an $(s \times s)$ stable (or Hurwitz) matrix (*i.e.* with all eigenvalues having negative real part), C is an $(c \times c)$ *central* matrix, (*i.e.* with all eigenvalues lying on the imaginary axis), and the functions ψ and μ are C^r functions vanishing at $(y, z) = (0, 0)$ together with all their first order derivatives. Suppose $\pi : V \to \mathbb{R}^s$ is a mapping satisfying $\pi(0) = 0$. The submanifold

$$W^c = \{(y, z) \in \mathbb{R}^s \times V : y = \pi(z)\}$$

is invariant under the flow of Eq. (4.2.5) if the mapping π satisfies the partial differential equation

$$\frac{\partial \pi}{\partial z}(Cz + \mu(\pi(z), z)) = S\pi(z) \ + \ Tz \ + \ \psi(\pi(z), z). \tag{4.2.6}$$

Comparing the first order terms on both sides of above, it is seen that the matrix

$$\Pi = \frac{\partial \pi}{\partial z}(0)$$

satisfies

$$S\Pi \ + \ T = \Pi C.$$

That is, Π satisfies

$$\begin{bmatrix} S & T \\ O & C \end{bmatrix} \begin{bmatrix} \Pi \\ I \end{bmatrix} = \begin{bmatrix} \Pi \\ I \end{bmatrix} C.$$

Thus, we deduce that

$$\text{Image}(\begin{bmatrix} \Pi \\ I \end{bmatrix}) = E^c$$

Thus, W^c is a center manifold for Eq. (4.2.5) if and only if Eq. (4.2.6) holds. □

The following lemma is a standard result in center manifold theory. This lemma shows that any trajectory of the system Eq. (4.2.2) starting at a point in a sufficiently small neighborhood of the origin of $\mathbb{R}^n$ converges to the center manifold as $t \to \infty$, with exponential decay.

Lemma 4.2.1 *Suppose $y = \pi(z)$ is a center manifold for Eq. (4.2.2) at $(0, 0)$. Let $(y(t), z(t))$ be a solution of Eq. (4.2.2). There exist a neighborhood U^0 of $(0, 0)$ and real numbers $M > 0$, $a > 0$ such that, if $(y(0), z(0)) \in U^0$, then*

$$\|y(t) - \pi(z(t))\| \ \leq \ M \exp(-at) \ \|y(0) - \pi(z(0))\|$$

for all $t \geq 0$, so long as $(y(t), z(t)) \in U^0$. □

The following theorem, known as the *reduction principle*, provides a more detailed picture about the role that the center manifold plays in the analysis of the asymptotic properties of the system Eq. (4.2.2) near $(0, 0)$. By definition, any trajectory of Eq. (4.2.2) starting at a point $y^0 = \pi(z^0)$ of the center manifold $y = \pi(z)$ can be described in the form

$$y(t) = \pi(z(t)), \quad z(t) = \zeta(t)$$

where $\zeta(t)$ is the solution of the differential equation

$$\dot{\zeta} = C\zeta + \mu(\pi(\zeta), \zeta) \tag{4.2.7}$$

satisfying the initial condition $\zeta(0) = z^0$. Essentially, the *reduction principle* shows that the asymptotic behavior of Eq. (4.2.2) is completely determined by that of Eq. (4.2.7).

Theorem 4.2.2 (**Reduction Principle**) *Suppose $\zeta = 0$ is a stable (resp. asymptotically stable, unstable) equilibrium of Eq. (4.2.7). Then $(y, z) = (0, 0)$ is a stable (resp. asymptotically stable, unstable) equilibrium of Eq. (4.2.2).* □

As an application of reduction principle, we have the following interesting lemma. (See also A. Isidori ([Isi89], pp. 442-443).

Lemma 4.2.2 *Consider a system*

$$\begin{aligned} \dot{z} &= f(z, y), \\ \dot{y} &= Ay + p(z, y), \end{aligned} \tag{4.2.8}$$

and suppose that $p(z, 0) = 0$ for all z near 0 and

$$\frac{\partial p}{\partial y}(0, 0) = 0. \tag{4.2.9}$$

If $\dot{z} = f(z, 0)$ has an asymptotically stable equilibrium at $z = 0$ and the eigenvalues of A all have negative real part, then the system (4.2.8) has an asymptotically stable equilibrium at $(z, y) = (0, 0)$. □

As another application of the reduction principle, we prove the following lemma.

Lemma 4.2.3 *Consider a triangular system*

$$\begin{aligned} \dot{x} &= f(x), \\ \dot{y} &= g(x, y) \end{aligned} \tag{4.2.10}$$

where $f : E \to \mathbb{R}^n$ is a locally C^1 map from a domain $E \subseteq \mathbb{R}^n$ into $\mathbb{R}^n$, and $g : E \to F \to \mathbb{R}^p$ is a locally C^1 map from a domain $E \subseteq F \subseteq \mathbb{R}^n \times \mathbb{R}^p$ into $\mathbb{R}^p$. Suppose that the system

$$\dot{x} = f(x) \tag{4.2.11}$$

is neutrally stable at the origin, and the system

$$\dot{y} = g(0, y) \tag{4.2.12}$$

is locally exponentially stable at the origin. Then the full system (4.2.10) is Lyapunov stable at the origin $(x, y) = (0, 0)$.

Proof. Linearizing the full system (4.2.10) at the origin $(x, y) = (0, 0)$, we have

$$\begin{aligned} \dot{x} &= Ax + \alpha(x), \\ \dot{y} &= By + \beta(x, y) \end{aligned} \tag{4.2.13}$$

where A is an $(n \times n)$ matrix having all its eigenvalues with zero real part, B is a $(p \times p)$ Hurwitz matrix, and the functions α and β vanish at the origin together with all their first partial derivatives.

By the center manifold theorem, the system (4.2.13) has a $\mathcal{C}^1$ center manifold at the origin, the graph of a $\mathcal{C}^1$ mapping

$$y = \pi(x)$$

with the map *pi* satisfying

$$\pi(0) = 0 \text{ and } D\pi(0) = 0.$$

The motion on the center manifold $y = \pi(x)$ is governed by the reduced dynamics

$$\dot{x} = f(x)$$

which, by hypothesis, is Lyapunov stable at the origin.

Hence, it follows immediately from the reduction principle that the full system (4.2.10) is Lyapunov stable at $(x, y) = (0, 0)$.

The reduction principle is very useful in stability theory, because it reduces the stability analysis of an n-dimensional differential system to that of a lower dimensional system (namely, c-dimensional) differential system. To implement the reduction principle in practice, we need to solve the center manifold equation

$$\frac{\partial \pi}{\partial x_1}(Cx_1 + \alpha(x_1, \pi(x_1))) = S\pi(x_1) + \beta(x_1, \pi(x_1))$$

which is quite difficult in general. However, it is always possible to approximate the solution $x_2 = \pi(x_1)$ of the center manifold equation to any required degree of accuracy. We can then use the approximate solution thus found in the reduced system (E.2.39). In this way, we may still determine the asymptotic properties of the equilibrium $x_1 = 0$ of the reduced system (E.2.39), and thus determine the asymptotic properties of the equilibrium $(x_1, x_2) = (0, 0)$ of the system (E.2.8).

The center manifold equation can be rewritten as

$$\mathcal{N}(\pi(x_1)) = \frac{\partial \pi}{\partial x_1}[Cx_1 + \alpha(x_1, \pi(x_1))] - S\pi(x_1)] - S\pi(x_1) - \beta(x_1, \pi(x_1)) = 0, \quad (4.2.14)$$

with the boundary conditions

$$\pi(0) = 0 \quad \text{and} \quad D\pi(0) = 0. \quad (4.2.15)$$

Theorem 4.2.3 *[Car81] Suppose that a function $\phi(x_1)$, with $\phi(0) = 0, D\phi(0) = 0$, can be found such that $\mathcal{N}(\phi(x_1)) = O(\|x_1\|^k)$ for some $k > 1$ as $\|x_1\| \to 0$. Then*

$$\pi(x_1) = \phi(x_1) + O(\|x_1\|^k), \quad \textit{as} \quad \|x_1\| \to 0. \ \square$$

Thus, we can approximate $\pi(x_1)$ to any required degree of accuracy by seeking power series solutions of (4.2.14). However, such power series expansions do not always exist, since W^c_{loc} may not be analytic at $(x_1, x_2) = (0, 0)$ (see Example E.2.3).

As an application of Theorem 4.2.3, we consider the following example.

Example 4.2.1 Consider the system

$\dot{x}_1 = -2x_1 + x_2^2$

$\dot{x}_2 = \epsilon x_1 x_2$

$$\begin{aligned} \dot{x}_1 &= \epsilon x_1 x_2, \\ \dot{x}_2 &= -2x_2 + x_1^2. \end{aligned} \quad (4.2.16)$$

The center manifold equation for the system (4.2.16) is

$x_1 = \pi(x_2)$

$$\frac{\partial \pi}{\partial x_1}[\epsilon x_1 \pi(x_1)] = -2\pi(x_1) + x_1^2, \quad (4.2.17)$$

with the boundary conditions

$\frac{\partial \pi}{\partial x_2}(\epsilon x_1 x_2)\Big|_{x_1 = \pi(x_2)}$

$$\pi(x_1) = 0 \quad \text{and} \quad D\pi(x_1) = 0. \quad (4.2.18)$$

$= -2\pi(x_1) + x_1^2$

$\epsilon \frac{\partial \pi}{\partial x_2} \pi(x_2) x_2 = -2\pi(x_2) + x_2^2$

$\pi(x_2) = 0 \quad \frac{\partial \pi}{\partial x_2}(0) = 0$

$2\epsilon a^2 x_2^3$

$x_1 = \pi(x_2) = a x_2^2 + O(\|x_2\|^3)$

$\epsilon\, 2a x_2\, a x_2^2$ $\qquad \frac{\partial \pi}{\partial x_2} = 2a x_2 + O(\|x_2\|)$

$\epsilon\, (2a x_2 + O(\|x\|^2))(a x_2^3 + O(\|x\|^4))$

$= -2a x_2^2 + O(\|x_2\|^3) + x_2^2$

The simplest approximation that we may try for $\pi(x_1)$ is

$$\pi(x_1) = ax_1^2 + O(|x_1|^3), \tag{4.2.19}$$

where a is a constant to be determined. Substituting (4.2.19) in (4.2.17) gives

$$(2ax_1 + O(|x_1|^2))(2a\epsilon x_1^3 + O(|x_1|^4)) = -2ax_1^2 + x_1^2 + O(|x_1|^3). \tag{4.2.20}$$

So, we have

$$-2a + 1 = 0 \quad \text{or} \quad a = \frac{1}{2}.$$

Thus, we may set

$$\pi(x_1) = \frac{1}{2}x_1^2 + O(|x_1|^3).$$

Hence, the reduced system is

$$\dot{x_1} = \epsilon x_1 \pi(x_1) = \frac{1}{2}\epsilon x_1^3 + O(|x_1|^4). \tag{4.2.21}$$

Using Lemma 3.1.1, we deduce that the equilibrium $x_1 = 0$ of the reduced system (4.2.21) is asymptotically stable if and only if $\epsilon < 0$. Using the reduction principle, we can then conclude that the equilibrium $(x_1, x_2) = (0, 0)$ of the system (4.2.16) is asymptotically stable if and only if $\epsilon < 0$. □

We end this section with a geometric proof of Malkin's theorem contained in [Hu89].

Theorem 4.2.4 (**Malkin's Theorem**) *Consider the system described by*

$$\begin{aligned} \dot{x_1} &= Rx_2 + p(x_1, x_2), \\ \dot{x_2} &= Sx_2 + q(x_1, x_2) \end{aligned} \tag{4.2.22}$$

where $x_1 \in \mathbb{R}^c, x_2 \in \mathbb{R}^s$, R, S are constant matrices, S is Hurwitz, p, q are analytic functions vanishing at $(x_1, x_2) = (0, 0)$ together with all their first order partial derivatives, and the Taylor series expansion of $q(x_1, 0)$ begins with terms of degree at least $N + 1$, where $N \geq 2$. If the equilibrium $x_1 = 0$ of the reduced system

$$\dot{x_1} = p(x_1, 0) \tag{4.2.23}$$

is stable, asymptotically stable, or unstable in the Nth approximation, then the equilibrium $(x_1, x_2) = (0, 0)$ of the full system (4.2.22) is stable, asymptotically stable, or unstable respectively.

Proof. [Hu89] Due to the term Rx_2, we can not apply the center manifold theorem directly. We first make a change of coordinates

$$\begin{bmatrix} \tilde{x_1} \\ \tilde{x_2} \end{bmatrix} = \begin{bmatrix} I & -RQ^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Then, under the new coordinates (here, we denote them by x_1 and x_2),

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} I & -RQ^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & R \\ 0 & Q \end{bmatrix} \begin{bmatrix} I & RQ^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} I & -RQ^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix},$$

i.e.

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} p(x_1 + RQ^{-1}x_2, x_2) + RQ^{-1}q(x_1 + RQ^{-1}x_2, x_2) \\ q(x_1 + RQ^{-1}x_2, x_2) \end{bmatrix}. \quad (4.2.24)$$

By the center manifold theorem, the system (4.2.24) has a center manifold at the origin, the graph of a $\mathcal{C}^1$ map, $x_2 = \pi(x_1)$ with π satisfying

$$\pi(0) = 0 \text{ and } D\pi(0) = 0.$$

By hypothesis, the expansion of $q(x_1, 0)$ begins with terms of degree $N + k$, where $k \geq 1$.

Our goal is to prove $\pi(x_1) = O(\|x_1\|^{N+k})$.

For this purpose, we consider the analytic function

$$F(x_1, x_2) = Qx_2 + q(x_1 + RQ^{-1}x_2, x_2).$$

Then, $F(0, 0) = 0$ and

$$\frac{\partial F}{\partial x_2}(0, 0) = Q.$$

Since Q is Hurwitz, it is nonsingular. Therefore, it follows from the implicit function theorem that there exist a $\delta > 0$ and a unique analytic function G such that, for $\|x_1\| < b$, we have

$$QG(x_1) + q(x_1 + RQ^{-1}G(x_1), G(x_1)) = 0. \quad (4.2.25)$$

Since the expansion of $q(x_1, 0)$ begins with terms of degree $N + k$, $q(0) = 0$, $Dq(0, 0) = 0$, and $G(x_1)$ is analytic, the expansion of $G(x_1)$ must also begin from terms of degree $N + k$.

Thus, we find that

$$\begin{aligned}(MG)(x_1) &\triangleq DG(x_1)[p(x_1 + RQ^{-1}G(x_1), G(x_1))] \\ &= O(\|x_1\|^{N+k-1}(O(\|x_1\|^2) + O(\|x_1\|^{N+k}) = O(\|x_1\|^{N+k+1}.\end{aligned}$$

Then, from Theorem 4.2.3, it follows that

$$\pi(x_1) = G(x_1) + O(\|x_1\|^{N+k+1}) = O(\|x_1\|^{N+k}).$$

The flow on the center manifold is governed by the equation

$$\dot{x_1} = p(x_1 + RQ^{-1}\pi(x_1), \pi(x_1)) + RQ^{-1}q(x_1 + RQ^{-1}\pi(x_1), \pi(x_1)),$$

i.e.

$$\dot{x_1} = p(x_1, 0) + O(\|x_1\|^{N+k}). \tag{4.2.26}$$

If the equilibrium $x_1 = 0$ of the dynamics

$$\dot{x_1} = p(x_1, 0) \tag{4.2.27}$$

is stable, asymptotically stable, or unstable in the Nth approximation, then it is also stable, asymptotically stable, or unstable for the system (4.2.26) respectively. Therefore, by the reduction principle, we conclude that if the equilibrium $x_1 = 0$ of the system (4.2.27) is stable, asymptotically stable, or unstable in the Nth approximation, then the equilibrium $(x_1, x_2) = (0, 0)$ is stable, asymptotically stable, or unstable for the full system (4.2.22) respectively. This completes the proof. □

4.3 Output Regulation of Nonlinear Systems

As an application of center manifold theory, we describe in this section the results contained in [AB90] solving the state feedback and error feedback regulator problems.

We consider a multivariable nonlinear plant described by

$$\begin{aligned}\dot{x} &= f(x) + g(x)u + p(x)w, \\ \dot{w} &= s(w), \\ e &= h(x) + q(w).\end{aligned} \tag{4.3.1}$$

The first equation of (4.3.1) describes, a *plant,* with state x, defined on a neighborhood X of the origin of $\mathbb{R}^n$, and input $u \in \mathbb{R}^m$. The evolution of the state x