Washington University in St. Louis

SCHOOL OF ENGINEERING & APPLIED SCIENCE

ESE 553 - Nonlinear Dynamical Systems

Christopher I. Byrnes

Ph. D., P.E.

Department of Electrical and Systems Engineering

Office: 421, Jolley Hall

Phone: 935-6067

E-mail: chrisbyrnes@wustl.edu

February 18 , 2008

Smooth functions on $\mathbf{R}^n$

Contraction Mapping Principle

Inverse Function Theorem

Implicit Function Theorem

Smooth functions on **R**^{n}

Contraction Mapping Principle

Inverse Function Theorem

Implicit Function Theorem

The Contraction Mapping Principle (Euclidean version). Let $C \subset \mathbb{R}^n$ be a closed subset and suppose $F : C \to C$ is a mapping satisfying

$$||F(x) - F(y)|| \le K||x - y||, \ \ 0 < K < 1.$$

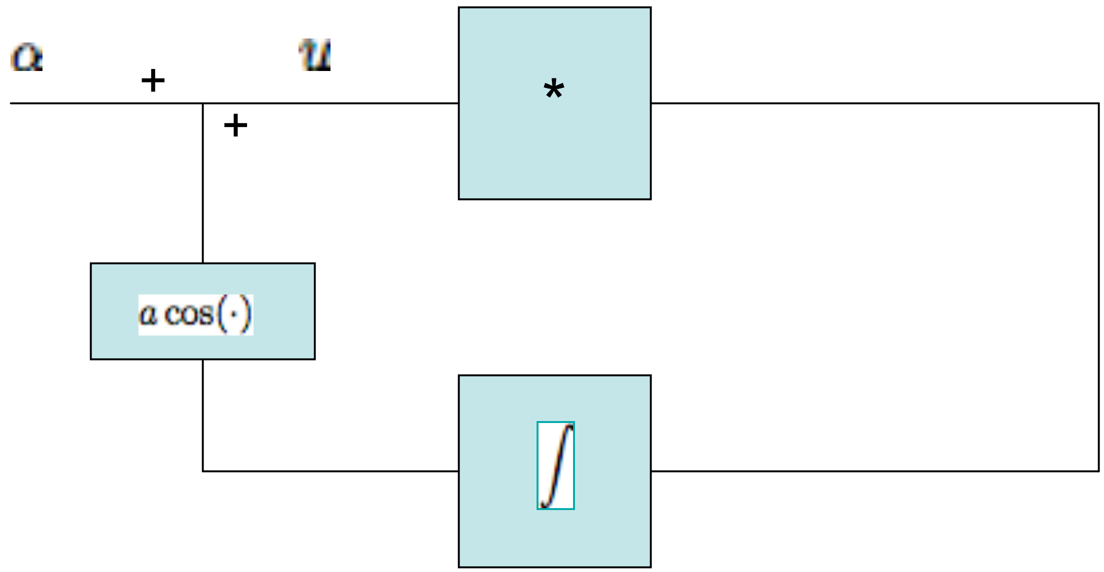
Then, there exists a unique $x_0 \in C$ such that $F(x_0) = x_0$. Moreover, for any $x \in C$ we have

$$||x_0 - F^n(x)|| \le K^n ||x_0 - x||$$

so that

$$\lim_{n \to \infty} F^n(x) = x_0$$

Example 1. VCO with a one dimensional loop filter.



$$* \qquad \dot{y} = -y + u$$

$$\dot{y} = -y + \alpha + a cos(\theta), \quad \text{where} \quad a > 0$$

$$\dot{\theta} = y$$

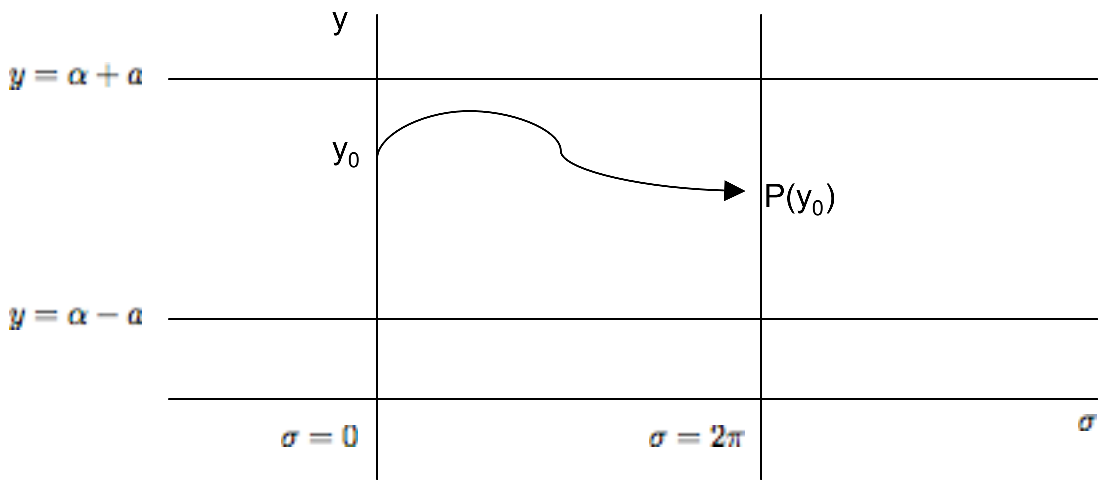
Example 1. (A Voltage Controlled Oscillator)

$$\dot{y} = -y + \alpha + a cos(\theta), \quad \text{where} \quad a > 0$$
$$\dot{\theta} = y$$

Lemma If $\alpha > a$, then $\mathcal{S} = \{(\theta, y) : \alpha - a < y < \alpha + a\}$ is positively invariant.

In particular, the VCO becomes:

$$y' = -1 + (\alpha + a\cos(\sigma))/y$$
$$\theta' = 1.$$



Example (A Voltage Controlled Oscillator as a 1 1/2 D System)

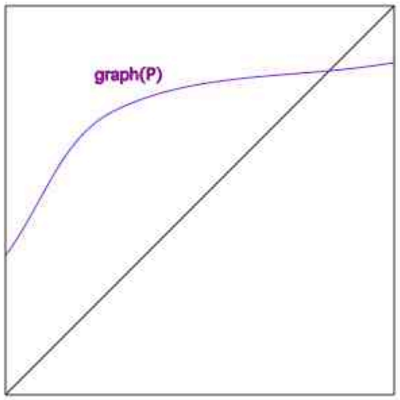
In particular, the VCO becomes:

$$y' = -1 + (\alpha + a\cos(\sigma))/y$$
$$\theta' = 1.$$

$$0 < \frac{dP}{dy}(y_0) = \mathrm{e}^{-\int_0^1 \frac{\alpha + a\cos(s)}{y(s,y_0)^2} ds} < 1$$

P is a contraction.

The VCO has a unique stable periodic orbit in S.



Example (A Voltage Controlled Oscillator as a 1 1/2 D System)

In particular, the VCO becomes:

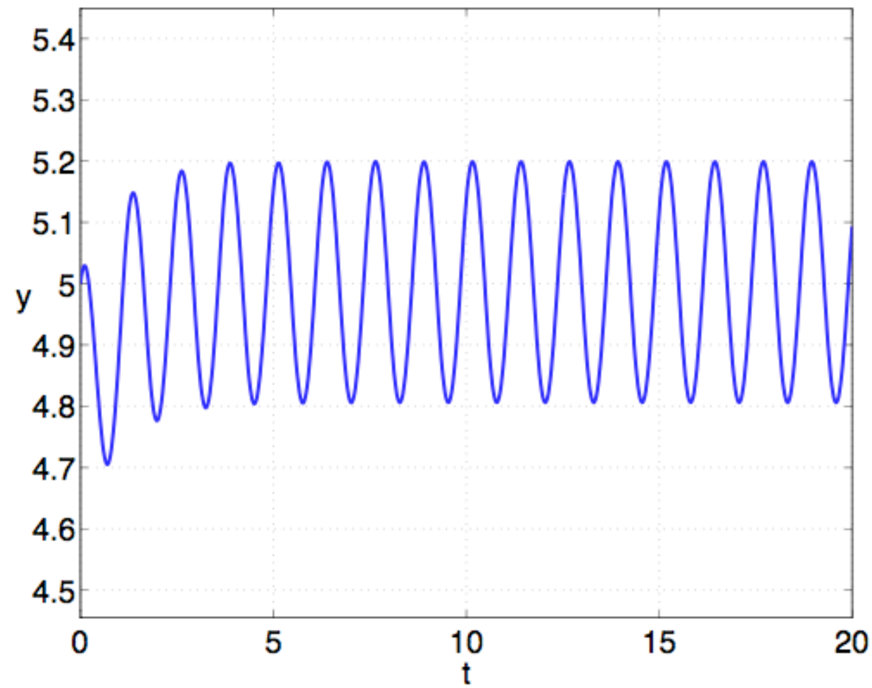
$$y' = -1 + (\alpha + a\cos(\sigma))/y$$
$$\theta' = 1.$$

$$0 < \frac{dP}{dy}(y_0) = e^{-\int_0^1 \frac{\alpha + a\cos(s)}{y(s,y_0)^2} ds} < 1$$

P is a contraction.

The VCO has a unique stable periodic orbit in S.

$$\alpha = 2, a = 1$$



Smooth functions on $\mathbf{R}^n$

Contraction Mapping Principle

Inverse Function Theorem

Implicit Function Theorem

The Inverse Function Theorem (Euclidean version). Suppose $U \subset \mathbb{R}^n$ is an open subset, and that $f : U \to \mathbb{R}^n$ is C^1 with a nonsingular Jacobian matrix $Df|_{x_0}$ for some $x_0 \in U$. Then, there exist an open set V, $x_0 \in V \subset U$, and an open set W, $f(x_0) \in W \subset \mathbb{R}^n$, such that f has an inverse function $g : W \to V$. Moreover, g is C^1 with Jacobian matrix:

$$Dg|_y = (Df|_{g(y)})^{-1}, \quad \text{for all } \ y \in W.$$

Smooth functions on $\mathbf{R}^n$

Contraction Mapping Principle

Inverse Function Theorem

Implicit Function Theorem

The Implicit function Theorem (Euclidean version). Let $U \subset \mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m$ be an open set and suppose $f : U \to \mathbb{R}^n$ is a C^1 function for which $D_x f|_{(a,b)}$ is invertible, for some point $(a,b) \in U$. Then, there exist an open set V, $(a,b) \in V \subset U$, and an open set W, $a \in W \subset \mathbb{R}^n$, and a *unique* C^1 function $g : W \to \mathbb{R}^n$ such that:

1. $g(b) = a$;
2. $f(g(u), u) = 0$;
3. $\{(x,u) \in V : f(x,u) = 0\} = \{(g(u), u) : u \in W\}$.

Moreover,

$$Dg|_u = -(D_x f)^{-1}|_{(g(u),u)} D_u f|_{(g(u),u)}$$

Smooth functions on **R**^{n}

Contraction Mapping Principle

Inverse Function Theorem

Implicit Function Theorem

N.B. The Implicit Function Theorem implies the Inverse Function Theorem.

Given $f : U \to \mathbb{R}^n$, define $F(x, u) = f(x) - u$ and note that $D_x F = D_x f$.

Therefore, the Implicit Function Theorem is a Fixed Point Theorem.

$C[0,1] = \{f : [0,1] \to \mathbb{R} : f \text{ is continuous}\}$

For $f \in C[0,1]$, define $\|f\| = sup_{x\in[0,1]}|f(x)|$.

1. $\|f\| = 0$ if and only if $f = 0$,
2. $\|\alpha f\| = |\alpha|\|f\|$ for any $\alpha \in \mathbb{R}$, and
3. $\|f + g\| \leq \|f\| + \|g\|$.

A sequence $(f_n) \in C[0,1]$ converges to $f \in C[0,1]$ if and only if

$$\|f_n - f\| = sup_{x\in[0,1]}|f_n(x) - f(x)| \to 0 \text{ as } n \to \infty.$$

N.B. Every Cauchy sequence in $C[0,1]$ converges.

The Contraction Mapping Principle ($C[0,1]$ version). Let $C \subset C[0,1]$ be a closed subset and suppose $F : C \to C$ is a mapping satisfying

$$\|F(x) - F(y)\| \leq K\|x - y\|, \quad 0 < K < 1.$$

Then, there exists a unique $x_0 \in C$ such that $F(x_0) = x_0$. Moreover, for any $x \in C$ we have

$$\|x_0 - F^n(x)\| \leq K^n\|x_0 - x\|$$

so that

$$\lim_{n\to\infty} F^n(x) = x_0$$

The Existence and Uniqueness theorem for ODE's. Consider the initial value problem $\dot{x} = f(x), x(0) = x_0$, where $x_0 \in \mathbb{R}^n$ and suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^1. Then there exists $a > 0$ and a unique solution

$$x : (-a, a) \to \mathbb{R}^n$$

of this ODE satisfying $x(0) = x_0$.

Remark. If y_0 and x_0 are two initial conditions, for which the solutions exist on a time interval $[0, T]$ then there exists a constant $K > 0$ such that

$$\|y(t) - x(t)\| \leq \mathrm{e}^{Kt} \|y_0 - x_0\|$$

for $t \in [0, T]$.

N.B. This is also true for parameters μ in $f(x, \mu)$.