Washington University in St. Louis

SCHOOL OF ENGINEERING & APPLIED SCIENCE

# ESE 553 - Nonlinear Dynamical Systems


Christopher I. Byrnes

**Ph. D., P.E.**

**Department of Electrical and Systems Engineering**

**Office: 421, Jolley Hall**

**Phone: 935-6067**

**E-mail: chrisbyrnes@wustl.edu**


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**The Poincaré-Bendixson Theorem.** If $\omega(x)$ is closed and bounded and does not contain an equilibrium, then $\omega(x) = \gamma$, a periodic orbit.

**Corollary.** If $\gamma$ is a limit cycle, then there exists an open set $U$ of initial conditions such that for $x_0 \in U$, $\omega(x_0) = \gamma$.

**Corollary.** If $U$ is a bounded positively invariant open set, there is a lower bound $T_U > 0$ on the period of any periodic orbit $\gamma \subset U$.

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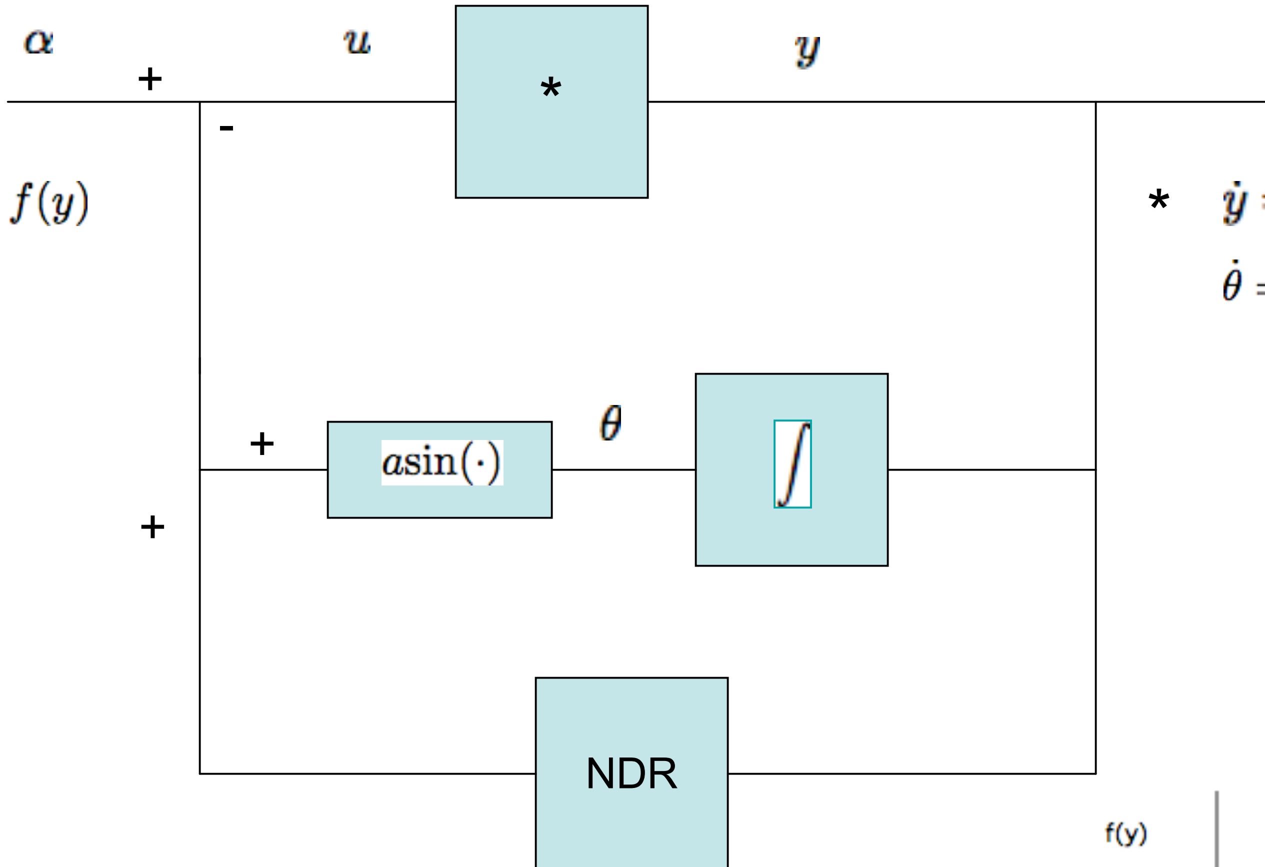
**Corollary.** If $\gamma$ is a periodic orbit and $U$ is the open set consisting of the bounded interior of $\gamma$, then there exists an equilibrium $x_0 \in U$.

# Example 1. (VCON: A Voltage Controlled Oscillator Neuron Model)

$u = \alpha - a\sin(\theta) - f(y)$

$a > 0$



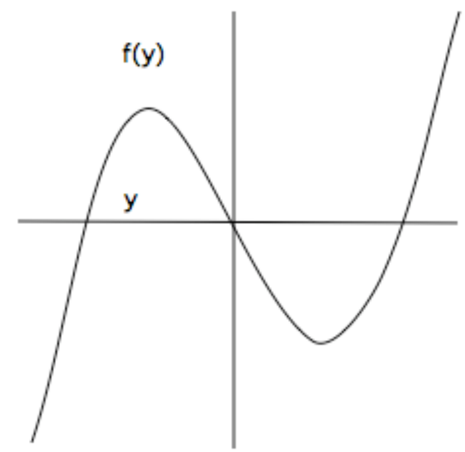

* $\dot{y} = -y + u$

$\dot{\theta} = y$

For $\theta \sim 0$ and

1. $f = 0$, classical PI controller
2. $f$ a NDR, van der Pol.

For $\theta$ large and $f = 0$, we have the classical *Voltage Controlled Oscillator*




A Nonlinear Differential Resistor (NDR)

**The van der Pol oscillator.** Consider the system

$$\dot{x} = y - x^3 + \mu x \quad (1)$$

$$\dot{y} = -x. \quad (2)$$

1. If $\mu < 0$, then the origin is globally asymptotically stable.

2. For $\mu = 0$, the linearized system is a harmonic oscillator, but the origin is globally asymptotically stable.

3. For $0 < \mu \leq 1$, the system has a periodic orbit $\gamma$ which is globally asymptotically stable in $\mathbb{R}^2 - \{(0,0)\}$.

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**Alternatively,**

$$\begin{aligned} \dot{x} &= y && (1) \\ \dot{y} &= -x + \lambda(1 - x^2)y && (2) \end{aligned}$$

Under the Lieńard transform, $u = x$, $v = y - \lambda(x - x^3/3)$ this becomes

$$\begin{aligned} \dot{u} &= v + \lambda(u - u^3/3) && (1) \\ \dot{v} &= -u && (2) \end{aligned}$$

**Alternatively,**

$$\dot{x} = y \quad (1)$$
$$\dot{y} = -x + \lambda(1 - x^2)y \quad (2)$$

1. If $\lambda < 0$, then the origin is globally asymptotically stable.
2. For $\lambda = 0$, the system is a harmonic oscillator.
3. For $0 < \lambda \leq 1$, the system has a periodic orbit $\gamma$ which is globally asymptotically stable in $\mathbb{R}^2 - \{(0,0)\}$.

# Stability of Periodic Orbits d'après Poincaré et Dulac

$$D\mathcal{P}(x_0) = \exp\left(\int_0^{t_f} \mathrm{div}(f)(\gamma(t))\, dt\right)$$

Poincaré: If the integral is negative, then $\gamma$ is stable.

Dulac: If there exists $\varphi : A^2 \to \mathbb{R}^+$ such that $\mathrm{div}(\varphi f) < 0$ on $A^2$ then there exists a unique cycle, which is asymptotically stable.

Example: (The van der Pol oscillator)

$$\left\{ \begin{array}{l} \dot{x} = y \\ \dot{y} = -x + \mu(1 - x^2)y \end{array} \right\}$$

The only equilibrium is at $(x, y) = (0, 0)$.

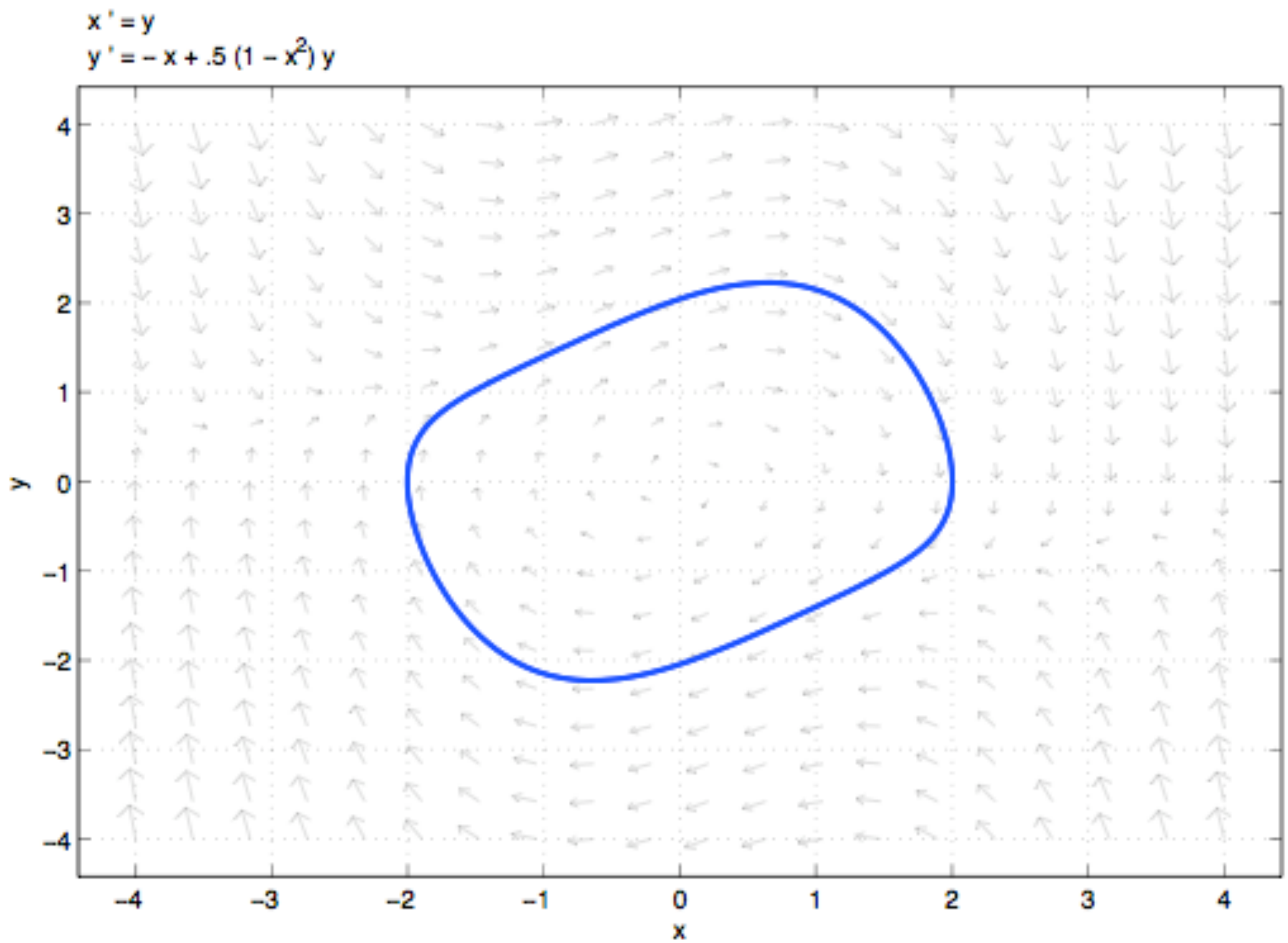
In fact, $\gamma$ is asymptotically stable. Set

$$\phi(x,y) = (x^2 + y^2 - \rho)^{-1/2}$$

Then,

$$\frac{\partial(\phi y)}{\partial x} + \frac{\partial{-\phi x + \lambda\phi(1-x^2)y)}}{\partial y} = -\frac{\lambda(x^2-\rho)^2}{(x^2+y^2-\lambda)^{3/2}} < 0$$

For $\rho = 1$ this is due to Cherkas.



x ' = y
y ' = − x + .5 (1 − x2) y
y
x
4
3
2
1
0
−1
−2
−3
−4
−4
−3
−2
−1
0
1
2
3
4

# Steady state behavior of nonlinear systems

For $B$ a bounded set, we define

$$\omega(B) = \{x | x = \lim_{t_n \to \infty} \phi(t_n, x_n) \ \text{ for } x_n \in B\}$$

Examples: The Lyapunov stable interval in the pitchfork bifurcation and the Lyapunov stable disc in the van der Pol oscillator.

The van der Pol oscillator is also undergoes a bifurcation at $\lambda = 0$. This is a complex conjugate pair of eigenvalues crossing the imaginary axis - a Hopf bifurcation.