



Washington University in St. Louis

SCHOOL OF ENGINEERING & APPLIED SCIENCE

ESE 553 - Nonlinear Dynamical Systems

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## Chapter 10.

- Flow Boxes
- Monotone Sequences
- The Poincare-Bendixson Theorem.

**The Poincaré-Bendixson Theorem.** If  $\omega(x)$  is closed and bounded and does not contain an equilibrium, then  $\omega(x) = \gamma$ , a periodic orbit.

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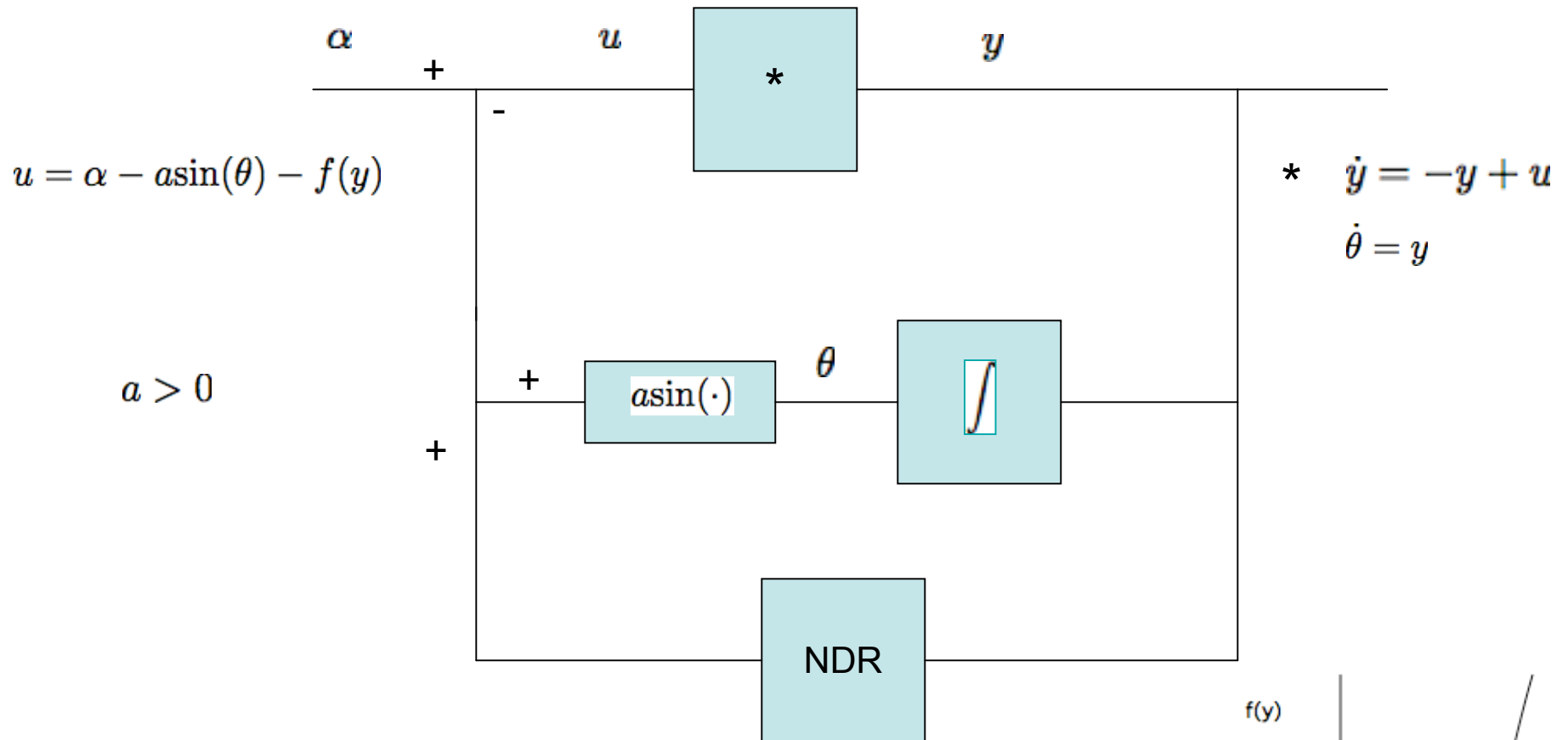
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**Corollary.** If  $\gamma$  is a periodic orbit and  $U$  is the open set consisting of the bounded interior of  $\gamma$ , then there exists an equilibrium  $x_0 \in U$ .

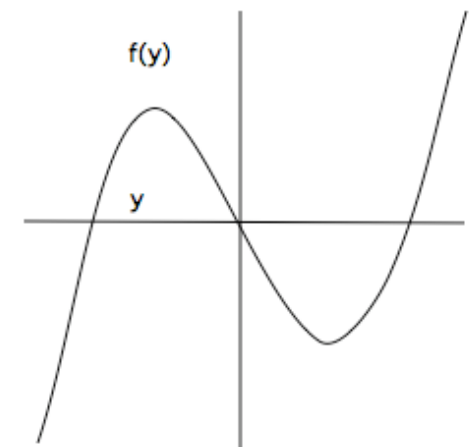
# Example 1. (VCON: A Voltage Controlled Oscillator Neuron Model)



For  $\theta \sim 0$  and

1.  $f = 0$ , classical PI controller
2.  $f$  a NDR, van der Pol.

For  $\theta$  large and  $f = 0$ , we have the classical *Voltage Controlled Oscillator*



A Nonlinear Differential Resistor (NDR)

**The van der Pol oscillator.** Consider the system

$$\dot{x} = y - x^3 + \mu x \tag{1}$$

$$\dot{y} = -x. \tag{2}$$

1. If  $\mu < 0$ , then the origin is globally asymptotically stable.
2. For  $\mu = 0$ , the linearized system is a harmonic oscillator, but the origin is globally asymptotically stable.
3. For  $0 < \mu \leq 1$ , the system has a periodic orbit  $\gamma$  which is globally asymptotically stable in  $\mathbb{R}^2 - \{(0, 0)\}$ .

## Stability of Periodic Orbits d'après Poincaré et Dulac

$$D\mathcal{P}(x_0) = \exp \left( \int_0^{t_f} \operatorname{div}(f)(\gamma(t)) dt \right)$$

Poincaré: If the integral is negative, then  $\gamma$  is stable.

Dulac: If there exists  $\varphi : A^2 \rightarrow \mathbb{R}^+$  such that  $\operatorname{div}(\varphi f) < 0$  on  $A^2$  then there exists a unique cycle, which is asymptotically stable.

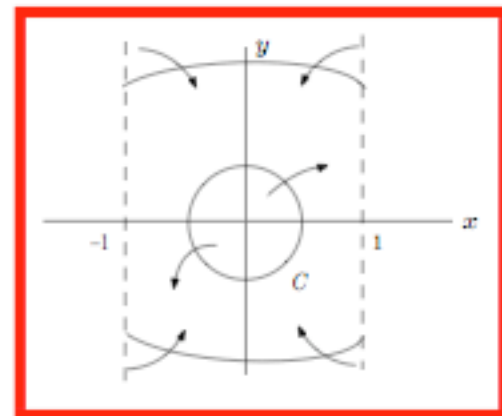
Example: (The van der Pol oscillator)

$$\left\{ \begin{array}{l} \dot{x} = y \\ \dot{y} = -x + \mu(1 - x^2)y \end{array} \right\}$$

The only equilibrium is at  $(x, y) = (0, 0)$ .

In particular  $|x(t)| \leq 1$  if  $|x(0)| \leq 1$ . Getting bounds on  $|y(t)|$  requires approximation methods, and perturbation methods, or studying cross sections and monotonicity of the Poincaré map.

Hale and Koçak construct, qualitatively, the outer ring of the Poincaré annulus (with corners). Therefore,  $\gamma$  exists.



In fact,  $\gamma$  is asymptotically orbitally stable and unique by Dulac's criterion:

Set  $\varphi(x, y) = (x^2 + y^2 - \lambda)^{-1/2}$ . Then

$$\frac{\partial}{\partial x}(\varphi y) + \frac{\partial}{\partial y}(-\varphi x + \mu\varphi(1 - x^2)y) = \frac{-\mu(x^2 - \lambda)^2}{(x^2 + y^2 - \lambda)^{3/2}} < 0$$

For  $\lambda = 1$ , due to Cherkas.