Washington University in St. Louis

SCHOOL OF ENGINEERING & APPLIED SCIENCE

ESE 553 - Nonlinear Dynamical Systems

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Stability of Periodic Orbits
- local sections
- the time to go
- the Poincare map
- Lyapunov stability
- an example

Consider the vector field $X \in \mathrm{Vect}(\mathbb{R}^3)$:

$$\dot{x}_1 = u_1 = -x_2 + (1 - x_3)x_1, \dot{x}_2 = u_2 = x_1 + (1 - x_3)x_2,$$

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3. On $\mathcal{I}$, X has a periodic orbit γ :

$$x(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \\ 1 \end{pmatrix}$$

with initial condition $x = (1, 0, 1)^T$

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4. γ attracts all points in $\mathbb{R}^3 - W^s(0)$.

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5. X and γ have a Lyapunov function

$$V(x_1, x_2, x_3) = (1 - x_3)^2 + (x_1^2 + x_2^2) - \ln(x_1^2 + x_2^2) - 1$$

defined except on $W^s(0)$ and achieving its minimum, 0, on γ.

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6. $\dot{V}(x_1, x_2, x_3) = -4(1 - x_3)^2 \leq 0$. The sublevel sets are positively invariant and $\mathrm{dist}(\phi(t, x_0), \gamma) \to 0$ as $t \to \infty$ for $x_0 \notin W^s(0)$.

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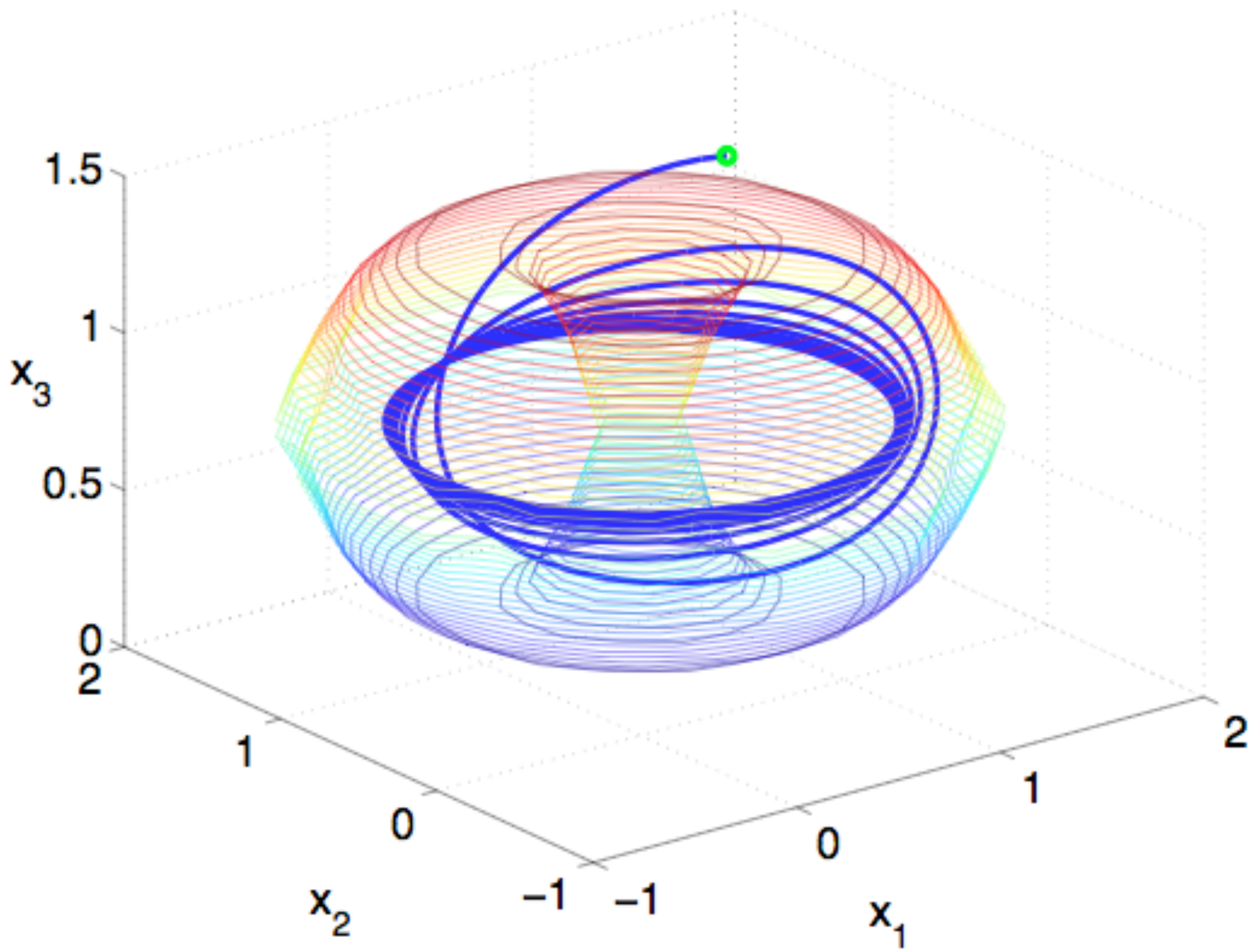
7. $D^2V|_\gamma \geq 0$, $\sigma(D^2V|_\gamma) = \{2, 4, 0\}$



The sublevel set $V^{-1}[0, 1]$,
with initial condition $(1, .75, 1.5)$.

Chapter 10.

- Flow Boxes
- Monotone Sequences
- The Poincare-Bendixson Theorem.