Washington University in St. Louis

School of Engineering & Applied Science

ESE 553 - Nonlinear Dynamical Systems

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The steady state response of forced, stable systems.

Theorem Consider the system $\dot{x} = f(x) + p(t, x)$ where $x \in U \subset \mathbb{R}^n$, f is C^1 and $0 = f(0)$ is an asymptotically stable equilibrium. for any $\epsilon > 0$ there exists positive $\delta_1(\epsilon), \delta_2(\epsilon)$ such that if

$|\xi_0\| < \delta_1(\epsilon), \;\; \|p(t, x)\|, \delta_2(\epsilon)$, then

$\|\phi(t, x_0)\| < \epsilon$ for $t \geq 0$.

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Theorem Consider $\dot{x} = f(x, w), \dot{w} = s(w)$, where $x \in \mathbb{R}^n, w \in \mathbb{R}^m$ and $f(0,0) = 0, s(0) = 0$. Suppose $x = 0$ and $w = 0$ are globally asymptotically stable equilibria for $\dot{x} = f(x, 0)$ and $\dot{w} = s(w)$ and the solutions of $\dot{x} = f(x, w)$ are bounded whenever w is bounded.

Then, $(x, w) = (0, 0)$ is globally asymptotically stable for the cascade system.

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Theorem Consider the system $\dot{x} = f(x) + p(t, x)$ where $x \in U \subset \mathbb{R}^n$, f is C^1 and $0 = f(0)$ is an asymptotically stable equilibrium. for any $\epsilon > 0$ there exists positive $\delta_1(\epsilon), \delta_2(\epsilon)$ such that if

$|\xi_0|| < \delta_1(\epsilon)$, $\|p(t, x)\|, \delta_2(\epsilon)$, then

$\|\phi(t, x_0)\| < \epsilon$ for $t \geq 0$.

Consider $\dot{x} = f(x) + g(x)u$ where 0 is asymptotically stable when $u = 0$. If $\|u\|$ is sufficiently small, then $\|\phi(t, x_0)\|$ is bounded when $\|x_0\|$ is small.

Examples $u = c$, or $u = A\sin(\omega t)$, where c or A is small.

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Consider $\dot{x} = f(x, u)$, where $f(0, 0) = 0$ and $\sigma(\frac{\partial f}{\partial x}|_{(0,0)}) \subset \mathbb{C}^-$.

If u is constant, there exists an asymptotically stable equilibrium $x = \pi(u)$ for U small.

This follows from either the Implicit Function Theorem or the Center Manifold Theorem:

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$$x = \pi(u), \ \dot{x} = \frac{\partial \pi}{\partial u}\dot{u}|_{x=\pi(u)} \implies f(\pi(u), u) = 0.$$

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Consider $\dot{x} = f(x, w)$, where $f(0,0) = 0$ and $\sigma(\frac{\partial f}{\partial x}|_{(0,0)}) \subset \mathbb{C}^-$.

Suppose $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ is a solution of the harmonic oscillator $\dot{w}_1 = w_2, \dot{w}_2 = -w_1$, with initial condition w_0.

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For $\epsilon_1 > 0$ there exists a smooth function π, $x = \pi(w)$, defined for $\|w_0\| < \epsilon_1$, such that graph$(\pi) \subset \mathbb{R}^n \times \mathbb{R}^2$ is invariant and exponentially attractive for the cascade system $\dot{x} = f(x, w), \dot{w}_1 = w_2, \dot{w}_2 = -w_1$

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For any $\|w_0\| < \epsilon_1$, the solution

$$\phi(t + 2\pi, \pi(w_0), w_0) = \phi(t, \pi(w_0), w_0) \text{ is periodic with period } 2\pi$$

Moreover, for any $\|(x_0, w_0)\| < \epsilon_2$, there is an asymptotic phase $\tau(x_0)$:

$$\|\phi(t, x_0, w_0) - \phi(t + \tau(x_0), \pi(w_0), w_0)\| \leq ke^{-\gamma t}\|(x_0, w_0)\|, \text{ for some } \gamma, k > 0.$$