Washington University in St. Louis

School of Engineering & Applied Science

ESE 553 - Nonlinear Dynamical Systems

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Analysis of some examples of nonlinear dynamics:

- 1 dimensional systems
- 2 dimensional systems
- 1 1/2 dimensional systems

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Finite dimensional (lumped) nonlinear dynamical systems:

$$x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad t \in \mathbb{R}, k \in \mathbb{Z}$$

Continuous time systems:	Discrete time systems:
$\dot{x} = f(x) \quad \textit{autonomous}$	$x_{k+1} = f(x_k), \quad \textit{autonomous}$
or	or
$\dot{x} = f(x,t) \quad \textit{nonautonomous}$	$x_{k+1} = f(x_k, k) \quad \textit{nonautonomous}$

In both cases, there is an initial time $t_0 \in \mathbb{R}$ or $t_0 \in \mathbb{Z}$, and an initial condition $x(t_0) \in \mathbb{R}^n$.

Finite dimensional nonlinear dynamical systems arising in bifurcations, forcing and feedback can have a particular form:

$$x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad t \in \mathbb{R}, k \in \mathbb{Z}$$

Continuous time systems:

$\dot{x} = f(x, t, u), \quad$ where

$u = u(x, t)$

Discrete time systems:

$x_{k+1} = f(x_k, k, u), \quad$ where

$u = u(x, k)$

In both cases, there is an initial time $t_0 \in \mathbb{R}$ or $t_0 \in \mathbb{Z}$, an initial condition $x(t_0) \in \mathbb{R}^n$ and $u = u(x, t)$.

u, for example, can be a constant parameter.

If $u = u(x)$, then we'll refer to u as a "feedback" term.
If $u = u(t)$ we'll refer to u as a "feedforward", or "forcing" term.

A population model with limits to growth and constant harvesting

$$\dot{x} = x(1-x) - h$$

Equilibrium equations:

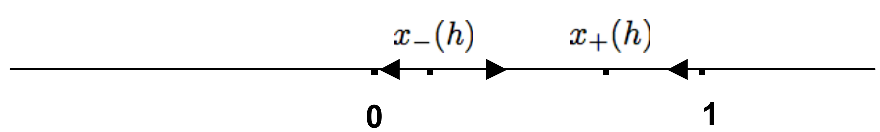
$$x^2 - x + h = 0$$

$$x_{\pm}(h) = \frac{1 \pm \sqrt{1-4h}}{2}$$

Equilibria exists for $0 \leq h \leq 1/4$

and satisfy: $0 \leq x_{\pm}(h) \leq 1$.

For $0 \leq h \leq 1/4$ and $|x| >> 0, \quad \dot{x} < 0$



N.B. **Extinction** occurs if h > 1/4. At h = 1/4, the dynamical system undergoes a **bifurcation**. To say h > 1/4 is to say, in our original model, that u > 4aN.

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Example

Consider the system

$$\dot{y} = ay - bky$$

$$\dot{k} = y^2$$

The equilibrium equations are:

$$ay - bky = 0$$

$$y^2 = 0$$

yielding a continuium of equilibria $(0, k)$.

Let's see how much $V(y) = \frac{y^2}{2}$ grows over time.

$$\dot{V} = y\dot{y} = ay^2 - by^2 = a\dot{k} - \frac{b}{2}(\dot{k^2})$$

or

$$\frac{d}{dt}(y^2 - ak + \frac{b}{2}k^2) = 0$$

so that $E(k,y) = y^2 - ak + \frac{b}{2}k^2$ is constant along trajectories.

In particular, trajectories evolve along an ellipse, with k increasing.

Therefore,

(1) all trajectories are bounded; and

(2) y(t) tends to 0 asymptotically.

Problem: Adaptive Stabilization

Consider the one-dimensional control system:

$\dot{y} = ay + bu$, where $y(0), a, b \in \mathbb{R}, b > 0$ are unknown.

Control Objective: Find a controller

$$\dot{z} = F(z, y), u = K(z, y)$$

such that for the closed-loop system

$$\dot{y} = ay + bK(z, y), \dot{z} = F(z, y)$$

(1) all solutions $y(t), z(t)$ exist and are bounded; and
(2) $\lim_{t\to\infty} y(t) = 0$.

Solution: On-line tuning of a classical feedback design

A simple candidate:

$$\dot{k} = y^2, u = -ky.$$

The closed-loop system is

$$\dot{y} = ay - bky$$
$$\dot{k} = y^2$$

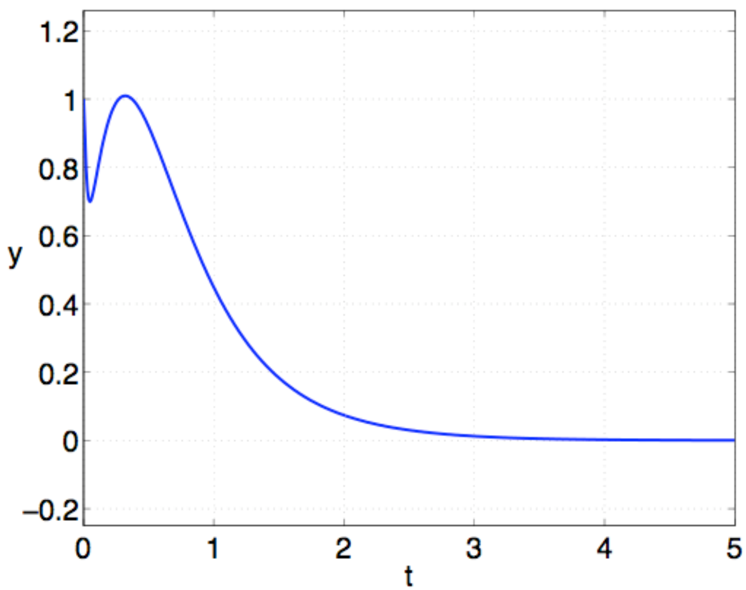
Since the trajectories evolve along an ellipse, with k increasing this simpler controller achieves the control objectives.

(1) all trajectories are bounded; and

(2) y(t) tends to 0 asymptotically.

Example (A Voltage Controlled Oscillator)

$$\dot{y} = -y + \alpha + acos(\theta), \quad \text{where} \quad a > 0$$
$$\dot{\theta} = y$$

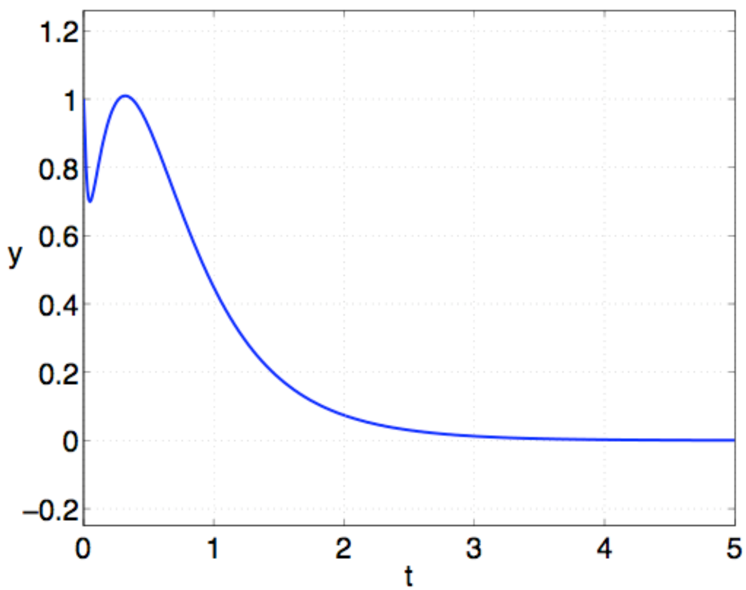


$\alpha = 1, a = 2$

Example (A Voltage Controlled Oscillator)

$$\dot{y} = -y + \alpha + a cos(\theta), \quad \text{where} \quad a > 0$$
$$\dot{\theta} = y$$

We claim this dynamical system has a periodic orbit of period T, for some $T > 0$ and some initial condition, whenever $\alpha > a$.



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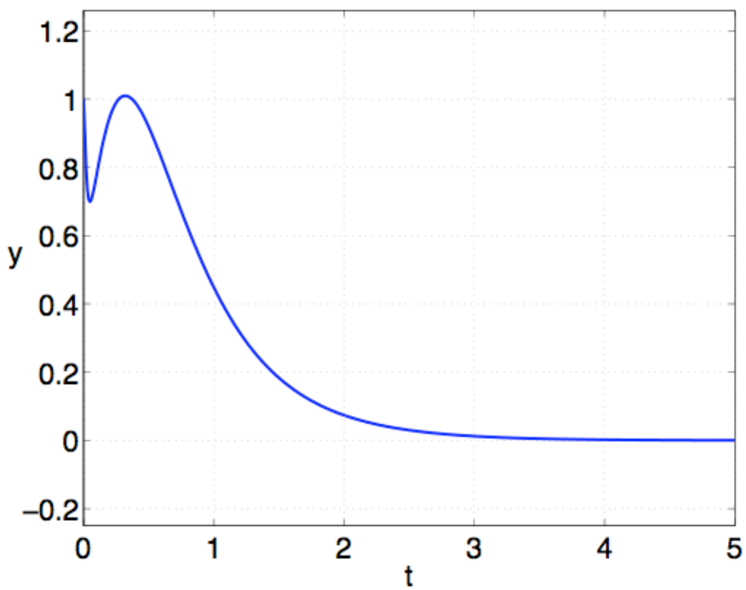

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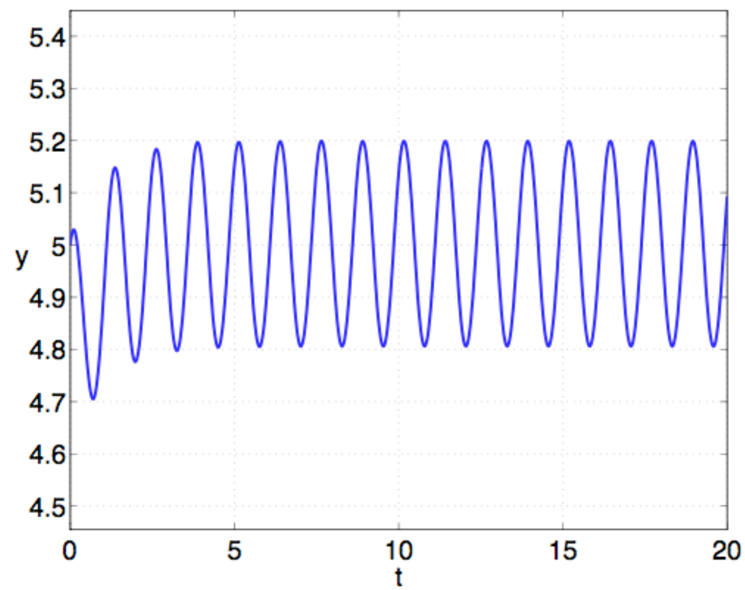

$\alpha = 2, a = 1$

Example (A Voltage Controlled Oscillator, VCO)

$$\dot{y} = -y + \alpha + acos(\theta), \quad \text{where} \quad a > 0$$

$$\dot{\theta} = y$$

Lemma For any initial condition $(\theta(0), y(0))$ a solution of the VCO equations exists for all $t \in \mathbb{R}$. Moreover, this solution is given by

$$\theta(t) = \theta(0) + \int_0^t y(\tau)d\tau, \;\; y(t) = \mathrm{e}^{-t}y(0) + \int_0^t \mathrm{e}^{-t+\tau} f(\theta(\tau)d\tau$$

where $f(\theta) = \alpha + a\cos(\theta)$.

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where $f(\theta) = \alpha + a\cos(\theta)$.

Caveat. For $\dot{x} = -x + x^2 + y(x+1), \;\; \dot{y} = -y + 1$ and the initial condition $x(0), y(0)) = (0, 1)$ a solution does not exist for all $t \in \mathbb{R}$, despite the stability of the linearization at $(0, 0)$.

Example (VCO)

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We claim this dynamical system has a periodic orbit of period T, for some $T > 0$ and some initial condition, whenever $\alpha > a$.

Equilibrium equations:

$-y + \alpha + acos(\theta) = 0, \quad y = 0$

$y = 0, \alpha + acos(\theta) = 0$

There only exists a solution if $\alpha \in [-a, a]$.

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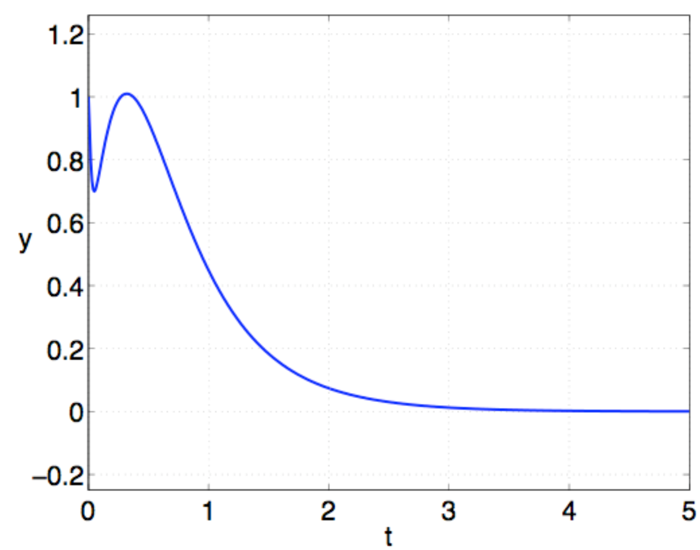

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Suppose a dynamical system $\dot{x} = f(x)$ evolves on $\mathbb{R}^n$. A subset $\mathcal{S} \subset \mathbb{R}^n$ is *positively invariant* if for any initial condition $x_0 \in \mathcal{S}$:

(1) the corresponding solution exists for all $t > 0$, and
(2) remains in $\mathcal{S}$.

$$\dot{y} = -y + \alpha + acos(\theta), \quad \text{where} \quad a > 0$$

$$\dot{\theta} = y$$

Lemma If $\alpha > a$, then $\mathcal{S} = \{(\theta, y) : \alpha - a < y < \alpha + a\}$ is positively invariant.