__Definition:__ (uniform convergence)

Let f_n and f be real-valued functions defined on an interval I. We say f_n converges uniformly to f on I if for every $\varepsilon > 0$ there exists an $N = N(\varepsilon)$ so that for all $n \geq N(\varepsilon)$ we have

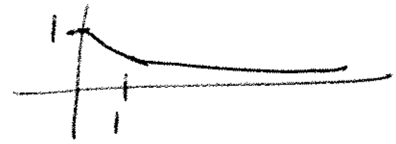
$$|f_n(x) - f(x)| < \varepsilon \qquad \text{for } \underline{\text{all } x \in I}.$$

__Remarks:__

- f_n converges uniformly to f on I if and only if $\{f_n\}$ is Cauchy uniformly over I,

$$|f_n(x) - f_m(x)| < \varepsilon \qquad \forall x \in I, \quad \forall n, m \geq N(\varepsilon).$$

__Examples:__ consider $f_n(x) = e^{-nx}$



(a) $I = [a, \infty)$ $\Rightarrow$ $|f_n(x)| = e^{-na} < \varepsilon \iff n \geq \frac{1}{a} \log\left(\frac{1}{\varepsilon}\right)$

converges uniformly to $f \equiv 0$ on every interval $[a, \infty)$ with $a > 0$

(b) but what if $a = 0$?

Proposition: let $f_n : I \to \mathbb{R}$ be a sequence of continuous functions that converges uniformly on I to some function f. Then f also is continuous on I.

pf. $|f(x) - f(x_0)| = |f(x) \mp f_n(x) \mp f_n(x_0) - f(x_0)|$

$$\leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|$$

given $\epsilon > 0$ there exists an $N = N(\epsilon)$ so that $|f_n(y) - f(y)| < \epsilon/3$ for all $n \geq N(\epsilon)$ and all $y \in I$. Pick $n = N(\epsilon)$. Since f_N is continuous, there exists a $\delta = \delta(\epsilon)$ so that $|f_N(x) - f_N(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$. Hence overall $|f(x) - f(x_0)| < \epsilon$ for $|x - x_0| < \delta$. ∎.

Ex. (6) revisited: $f_n(x) = e^{-nx}$ does NOT converge uniformly on $[1, \infty)$

e.g. $f_n(x) = x^n$ does not converge uniformly on $[0,1]$.

Definition: (metric space)

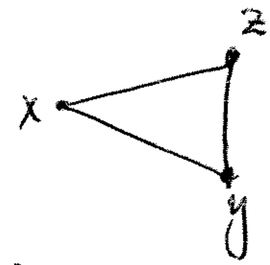
Let X be a vector space. A metric d on X is a function $d: X \times X \to [0,\infty)$, $(x,y) \mapsto d(x,y)$ such that

(a) positive definite: $d(x,y) = 0 \iff x = y$

(b) symmetric: $d(x,y) = d(y,x)$ for all $x, y \in X$

(c) triangle inequality: for all $x, y, z \in X$

$$d(x,z) \leq d(x,y) + d(y,z)$$



Examples:

(a) silly/trivial metric: $d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$

(b) ~~metric~~ $d(x,y) = |x-y|$ on $\mathbb{R}$

(c) $d(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$ on $\mathbb{R}^n$ (d) $d(x,y) = \|x-y\|$

translation invariant

Definition: (normed space)

Let X be a vector space. A norm $\|\cdot\|$ on X is a function $\|\cdot\|: X \to [0,\infty)$ so that (a) $\|x\| = 0 \iff x = 0$,

(b) $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in \mathbb{R}$, $x \in X$

(c) $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

a metric defines neighborhoods and thus a <u>topology</u> on X:

$$B_\varepsilon(x_0) = \{ x \in X : d(x, x_0) < \varepsilon \} \quad \text{- open "ball"}$$

$$x_n \xrightarrow{d} x \quad :\Leftrightarrow \quad d(x_n, x) \to 0 \text{ in } \mathbb{R} \Rightarrow \text{continuity}$$

a set $E \subseteq X$ is open if for every $x_0 \in E$ there exists an $\varepsilon > 0$ so that $B_\varepsilon(x_0) \subseteq E$; complements of open sets are closed.

<u>Lemma:</u> a set $F \subseteq X$ is closed if and only if whenever $\{x_n\} \subseteq F$ is a sequence of points from F that converges to some limit $x \in X$, then necessarily $x \in F$.

<u>pf.</u> "$\Rightarrow$" suppose F is closed and $x \notin F$. Then there exists an $\varepsilon > 0$ so that $B_\varepsilon(x) \cap F = \emptyset$. but for n large enough $x_n \in B_\varepsilon(x)$ ↯.

"$\Leftarrow$" suppose all limits of sequences from F lie in F. If F^c is not open then there exist a point $y \in F^c$ so that every neighborhood $B_{\frac{1}{n}}(y)$ contains a point $x_n \in F$. But then $x_n \to y$ and so $y \in F$ ↯ ∎

<u>Definition:</u> (<u>completeness</u>)

Let (X, d) be a metric space. We say a sequence $\{x_n\}_{n \in \mathbb{N}} \subset X$ is Cauchy if for every $\varepsilon > 0$ there exists an $N = N(\varepsilon)$ so that for all $n, m \geq N(\varepsilon)$ $d(x_n, x_m) < \varepsilon$. A metric space is said to be complete if every Cauchy sequence converges.

<u>Example:</u> let $C([a,b])$ denote the normed space of all continuous functions $f:[a,b]\to\mathbb{R}$ with the norm

$$\|f\| = \max_{a\le t\le b} |f(t)| \quad . \text{ Then } C([a,b]) \text{ is complete.}$$

pf. Suppose $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy, i.e.

$$\forall_{\varepsilon>0} \quad \exists_{N=N(\varepsilon)} \quad \forall_{n,m\ge N(\varepsilon)} \quad \forall_{t\in[a,b]} : |f_n(t)-f_m(t)| < \varepsilon$$

Then the sequence $\{f_n(t)\}_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$ for each fixed $t\in[a,b]$ and thus has a limit, call it $f(t)$. But then we also have

$$\forall_{\varepsilon>0} \quad \exists_{N=N(\varepsilon)} \quad \forall_{n\ge N(\varepsilon)} \quad \forall_{t\in[a,b]} : |f_n(t)-f(t)| \le \varepsilon$$

Since ε is arbitrary, f_n converges to f uniformly on $[a,b]$, hence f is continuous. We still need to convince ourselves that convergence is in the metric. Clearly, for $n\ge N(\varepsilon)$ we have

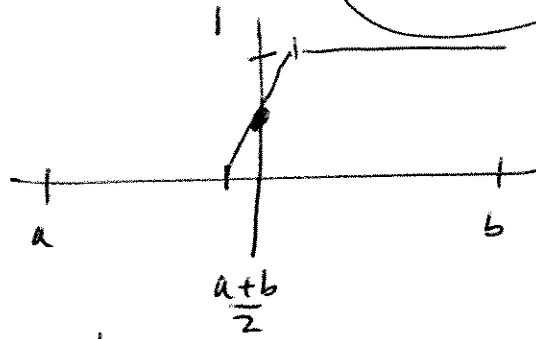
$$\|f_n-f\| = \max_{a\le t\le b} |f_n(t)-f(t)| \le \varepsilon \qquad \checkmark$$

$\boxtimes$

Example: let X be the space of continuous functions on $[a,b]$ and define $\|f\| = \left(\int_a^b |f(t)|^2 dt \right)^{1/2}$. define

is a norm!

$f_n(t) \rightarrow$



$$\begin{cases} 1 & \text{if } t \geq \frac{a+b}{2} + \frac{1}{n} \\ \text{linear in between} & \\ 0 & \text{if } t < \frac{a+b}{2} - \frac{1}{n} \end{cases}$$

$$f(t) = \begin{cases} 1 & \text{if } t > \frac{a+b}{2} \\ \frac{1}{2} & \text{if } t = \frac{1}{2}(a+b) \\ 0 & \text{if } t < \frac{1}{2}(a+b) \end{cases}$$

verify that $\|f_n - f\| \to 0$.

So this space is <u>NOT</u> complete

<u>fact:</u> given any metric space (X, d), there exists a unique complete metric space $(\hat{X}, \hat{d})$ so that $\hat{X} = \overline{X} = \operatorname{Clos} X$ and $\hat{d}(x,y) = d(x,y)$ for all $x, y \in X$: <u>completion</u>

$\hookrightarrow$ space $L^2([a,b])$ of Lebesgue integrable functions

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Definition: (contraction)

Let (X,d) be a metric space and $A \subseteq X$ a subset of X. We say a mapping $F: X \to X$ is a contraction on A if $F(A) \subseteq A$ and there exists a constant $K \in (0,1)$ so that

$$d(F(x), F(y)) \leq K\, d(x,y) \quad \text{for all } x,y \in A.$$

Theorem: Banach - fixed point theorem

Let (X,d) be a complete metric space and let F be a contraction on a closed subset A. Then the mapping F has a unique fixed point on A, $F(z) = z$, and z is the limit of any sequence of the form $x_{n+1} = F(x_n)$ with $x_0 \in A$ arbitrary.

pf. Pick $x_0 \in A$ arbitrary and define $\{x_n\}_{n \in \mathbb{N}}$ by $x_{n+1} = F(x_n)$. Since F is a contraction on A, the sequence lies in A and for any $n \geq m$

$$d(x_n, x_m) = d(F(x_{n-1}), F(x_{m-1})) \leq K\, d(x_{n-1}, x_{m-1}) \leq \dots$$

$$\leq K^m\, d(x_{n-m}, x_0)$$

$$\leq K^m \left[d(x_{n-m}, x_{n-m-1}) + d(x_{n-m-1}, x_{n-m-2}) + \dots + d(x_1, x_0) \right]$$

$$\leq K^m \left[K^{n-m-1} + K^{n-m-2} + \dots + K + 1 \right] d(x_1, x_0)$$

$$\leq \frac{K^m}{1-K} \, d(x_1, x_0)$$ and this can be made arbitrarily small for m large

So the sequence $\{x_n\}_{n \in \mathbb{N}}$ is Cauchy and thus has a limit $\bar{x} \in A$.

It is clear that contractions are continuous maps: if $x_n \to \bar{x}$, then

$$d(F(x_n), F(\bar{x})) \leq K \, d(x_n, \bar{x}) \to 0.$$

hence $F(\bar{x}) = \lim_{n\to\infty} F(x_n)$.

but by definition of the sequence

$$F(\bar{x}) = \lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} x_{n+1} = \bar{x},$$ so $\bar{x}$ is a fixed pt.

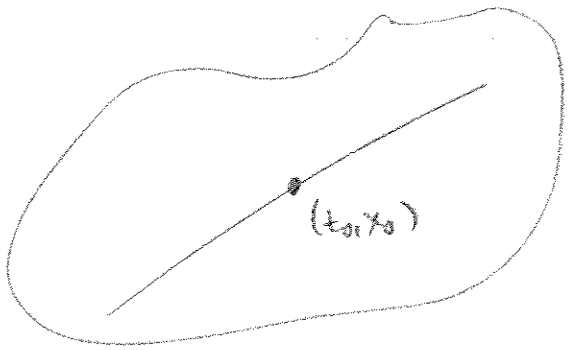
uniqueness: suppose $\bar{x}$ and $\bar{y}$ are both fixed points for F in A

then

$$\left. \begin{array}{l} \underbrace{d(F(\bar{x}), F(\bar{y}))}_{= d(\bar{x}, \bar{y})} \leq K \, d(\bar{x}, \bar{y}) \end{array} \right\} \Rightarrow d(\bar{x}, \bar{y}) = 0$$

Q.E.D.

Example: existence and uniqueness of solutions to ODE



Let $D \subseteq \mathbb{R} \times \mathbb{R}^n$ be an open set and suppose $f = f(t,x)$ is continuous and Lipschitz continuous in x on D.

$$\dot{x} = f(t,x), \quad x(t_0) = x_0 \quad \Leftrightarrow \quad x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds =: F(x)(t$$

define an operator $F : C([a,b]) \to C([a,b])$ by $x \mapsto F(x)$ given as

$$\underbrace{F(x)(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds} \qquad t_0 \in (a,b)$$

then x is a solution to the initial value problem if and only if x is a fixed point for the operator F.

$C([a,b])$ is a complete metric space $\Rightarrow$ we need to find a closed subset A on which F is a contraction.

This will be the case if we <u>choose the interval $[a,b]$ small enough</u>

Let $Q = [a,b] \times \overline{B_\varepsilon(x_0)}$, compact, and suppose f satisfies a Lipschitz condition on Q in the form

$$\| f(t,x) - f(t,y) \| \le L \| x-y \| \qquad x,y \in \overline{B_\varepsilon(x_0)}, \; t \in [a,b]$$

with $\|\cdot\|$ the Supremumsnorm on $\mathbb{R}^n$,

$\|x\| = \max_{i=1,\dots,n} |x_i|$. Since f is continuous, it is also bounded on Q, say $\| f(t,x) \| \le M < \infty$ for all $(t,x) \in Q$.

Let $A = \{ x \in C([a,b]) : x(t_0) = x_0, \; \|x(t) - x_0\| \le \varepsilon$ for all $t \in [a,b] \}$

(a) A is closed and $F(A) \subseteq A$:

$$\|F(x)(t) - x_0\| = \left\| \int_{t_0}^{t} f(s,x(s))\, ds \right\| \le \left| \int_{t_0}^{t} M\, ds \right| \le M(b-a)$$

choose $[a,b]$ so that $(b-a)M \le \varepsilon$ to get $F(A) \subseteq A$.

If $\{x_n\}$ is any sequence from A that converges uniformly on $[a,b]$, then also the limit $\bar{x}$ satisfies $\bar{x}(t_0) = x_0$ and

$$\|\bar{x}(t) - x_0\| = \lim_{n\to\infty} \|x_n(t) - x_0\| \le \varepsilon, \qquad \text{ie } \bar{x} \in A.$$

(b) F is a contraction on A:

$$\| F(x)(t) - F(y)(t) \| = \| \int_{t_0}^{t} f(s, x(s)) - f(s, y(s))\, ds \|$$

$$\leq \left| \int_{t_0}^{t} \| f(s, x(s)) - f(s, y(s)) \| \, ds \right|$$

$$\leq \left| \int_{t_0}^{t} L \| x(s) - y(s) \| \, ds \right| \leq (b-a) L \| x - y \|$$

thus, after starting with an initial cube $Q = [a,b] \times \overline{B_\varepsilon(x_0)}$ that lies in D and determines the constants M and L, simply make the interval smaller so that $(b-a) M \leq \varepsilon$ and $(b-a) L < 1$.

then F is a contraction on A and there exists a unique fixed point. This fixed point is a solution to the (IVP) $\dot{x} = f(t, x)$, $x(t_0) = x_0$, that exists on the full interval $[a,b]$.