Washington University in St. Louis

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ESE 553 - Nonlinear Dynamical Systems

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NONLINEAR SYSTEMS
Dynamical systems and differential equations.

A smooth dynamical systems on $\mathbb{R}^n$ is a C^1 map:

$\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ where $\phi_t(x) = \phi(t, x)$ satisfies:

1. $\phi_0(x) = x$ for all $x \in \mathbb{R}^n$.
2. $\phi_t \circ \phi_s = \phi_{s+t}$ for $t, s \in \mathbb{R}$.

The Existence and Uniqueness theorem for ODE's. Consider the initial value problem $\dot{x} = f(x), x(0) = x_0$, where $x_0 \in \mathbb{R}^n$ and suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^1. Then there exists $a > 0$ and a unique solution

$$x : (-a, a) \to \mathbb{R}^n$$

of this ODE satisfying $x(0) = x_0$.

Remark. If y_0 and x_0 are two initial conditions, for which the solutions exist on a time interval $[0, T]$ then there exists a constant $K > 0$ such that

$$\|y(t) - x(t)\| \leq \mathrm{e}^{Kt}\|y_0 - x_0\|$$

for $t \in [0, T]$.

N.B. This is also true for parameters μ in $f(x, \mu)$.

Lyapunov's Indirect Method Suppose $\dot{x} = f(x)$, where f is C^1. Suppose $f(0) = 0$, and $f(x) = Ax + R(x)$.

1. If $\sigma(A) \subset \mathbb{C}^-$ then 0 is a locally asymptotically stable equilibrium.

2. If $\sigma(A) \cap \mathbb{C}^+ \neq \emptyset$ then 0 is an unstable equilibrium.

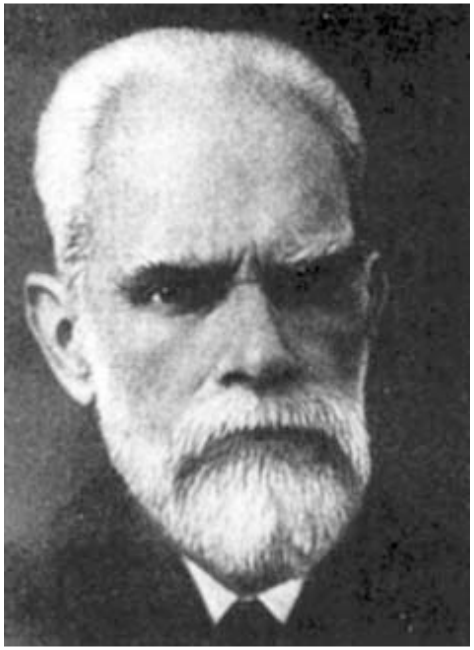
Theorem (Lyapunov's Direct Method) Suppose $\dot{x} = f(x)$, where $f(0) = 0$ and f is C^1 on an open set $0 \in \mathcal{O}$. Let $V : \mathcal{O} \to \mathbb{R}$ be a C^1 function such that

1. $V(0) = 0$ and $V(x) > 0$ if $x \neq 0$;
2. $\dot{V} \leq 0$ in $\mathcal{O}$.

Then 0 is stable. Furthermore, if V also satisfies

3. $\dot{V} < 0$ in $\mathcal{O} - 0$,

then 0 is asymptotically stable.



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Definition (Birkhoff) Suppose $\dot{x} = f(x)$, where f is C^1 on an open set $\mathcal{O}$. If x_0 is an initial condition and

$$\bar{x} = lim_{t_n \to \infty} \phi(t_n, x_0)$$

then $\bar{x}$ is an ω-limit point of x_0. If $t_n \to -\infty$ then $\bar{x}$ is an α-limit. The set of such points is denoted by $\omega(x_0)$ and $\alpha(x_0)$.

G. D. Birkhoff