Delta Sequence

Our objective is to provide a rigorous interpretation of the "delta function" suitable for understanding by calculus level students. In particular we want to make sense out of the formula

$$\int_{\mathbb{R}} \varphi(x)\, \delta_{x_0}(x)\, dx = \varphi(x_0) \tag{1}$$

where φ is a continuous function.

Define $f_n(x)$ by

$$f_n(x) = \begin{cases} n, & x_0 < x < x_0 + 1/n \\ 0, & \text{otherwise} \end{cases}$$

Then for any continuous φ we have

$$\int_{\mathbb{R}} \varphi(x)\, f_n(x)\, dx = n \int_{x_0}^{x_0+1/n} \varphi(x)\, dx. \tag{2}$$

Now, for any $c \in \mathbb{R}$, let us define

$$F(x) = \int_c^x \varphi(x)\, dx$$

then, by the fundamental theorem of calculus, we have

$$F'(x) = \varphi(x).$$

Next we notice that (2) can be written as the difference quotient for F,

$$\frac{F(x_0 + 1/n) - F(x_0)}{1/n},$$

and

$$\lim_{n\to\infty} \frac{F(x_0 + 1/n) - F(x_0)}{1/n} = F'(x_0) = \varphi(x_0). \tag{3}$$

Thus to make sense of the formula (1) we simply define

$$\int_{\mathbb{R}} \varphi(x)\, \delta_{x_0}(x)\, dx = \lim_{n\to\infty} \int_{\mathbb{R}} \varphi(x)\, f_n(x)\, dx = \varphi(x_0). \tag{4}$$

This is formula (1).