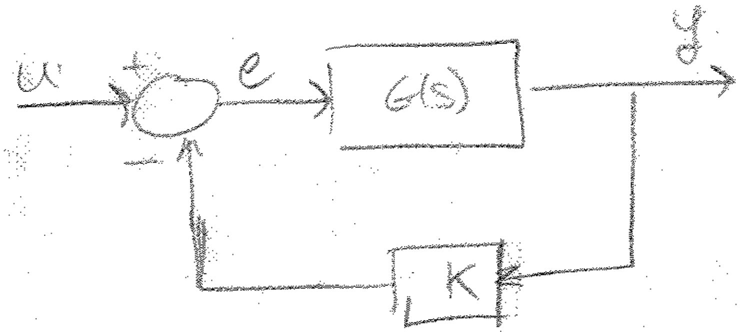
Root Locus

Consider the feedback system



where $G(s) = \frac{n(s)}{d(s)}$ and $\deg(n(s)) < \deg(d(s))$

The closed-loop transfer function for this system is given by:

$$H(s) = \frac{G(s)}{1 + KG(s)} = \frac{n(s)/d(s)}{1 + K\, n(s)/d(s)}$$

$$= \frac{n(s)}{d(s) + Kn(s)}$$

We are intrested in how K affects the CL poles as K varies from 0 to ∞.

i.e. the CL-poles are the roots of the equation

$$\boxed{d(s) + K n(s) = 0}$$

Def. The locus of these roots plotted in the complex plane $\mathbb{C}$ as a function of K is called the <u>root-locus</u>, and $0 < K < \infty$

$$d(s) + K n(s) = 0$$

If $K = 0$, the equation becomes $d(s) = 0$.

$\therefore$ The locus starts at the <u>poles of $G(s)$</u>

Note that we have $\frac{1}{K} d(s) + n(s) = 0$

Then as $K \to \infty$, the equation becomes $n(s) = 0$.

$\therefore$ The locus ends at the <u>zeros of $G(s)$</u>.

Rules for Graphical Construction

1) OL Poles and Zeros (assume n poles, m zeros, $n>m$)
Mark the n OL poles with an "x"
Mark the m OL zeros with an "o"

2) Branches of the Root Locus:
The root locus has exactly n branches, each branch. of these, m branches will approach the m OL zeros as $k \to \infty$.
The remaining $n-m$ branches will go to infinity.
(Note: n branches b/c it is the locus of a polynomial of degree $\deg(d(s))=n$ as a parameter, k, varies)

3) Asymptotic Rays:
The $n-m$ branches going to infinity asymptotically approach $n-m$ rays originating at a single point on the real axis. These rays are the asymptotes and their common origin is the center of the asymptotes.

The angle of the asymptotes are given by

$$\alpha = \frac{2l+1}{n-m}\pi \quad , \quad l=0,1,2,\ldots,n-m-1$$

i.e.

$n-m$	α
1	π
2	$\pm \pi/2$
3	$\pm \pi/3, \pi$
4	$\pm \pi/4, \pm 3\pi/4$

The center of the asymptotes is given by

$$\sigma_c = \frac{\sum p_j - \sum z_i}{n-m}$$

p_j = poles
z_i = zeros
of OL system

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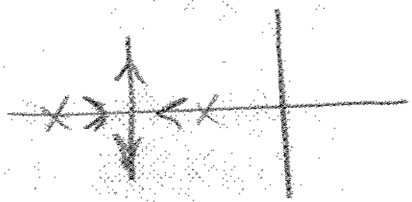
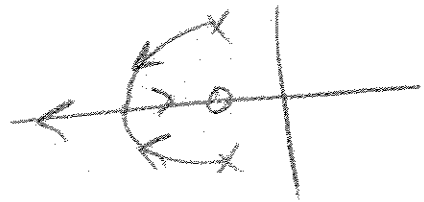
4) Real Part of the Root Locus

The part of the real axis to the left of an odd number of poles or zeros belong to the root locus.

5) <u>Breakaway Points</u>:

These are points at which two (or more) branches of the root-locus depart or arrive at the real axis





How to compute a breakaway point?

Recall, $d(s) + K\,n(s) = 0$.

Then $K(s) = -\dfrac{d(s)}{n(s)}$.

a breakaway point, σ_b, $K(s)$ obtains a maximum.

Thus, to find σ_b, we need to solve,

$$\frac{dK(s)}{ds} = 0.$$

Alternative approach: solve the equation.

$$\sum_{i=1}^{n} \frac{1}{(\sigma_b + p_i)} = \sum_{i=1}^{m} \frac{1}{(\sigma_b + z_i)}$$

c) Crossover Points:

These are the points where the branches of the root locus cross the imaginary axis

⇒ found by Routh-Hurwitz

Example: Plot the root locus for

$$G(s) = \frac{1}{(s+1)(s+2)(s+3)} = \frac{n(s)}{d(s)}.$$

The CL pole equation is $d(s) + K\,n(s) = 0$:

$$\boxed{s^3 + 6s^2 + 11s + (6 + K) = 0.}$$

OL poles are: $\boxed{p_1 = -1,\ p_2 = -2,\ p_3 = -3}$

No OL zeros

$$\Rightarrow n = 3,\ m = 0 \Rightarrow n - m = 3.$$

∴ The RL has 3 branches, all going to infinity along the asymptotes centred at the breakaway point.

$$\sigma_c = \frac{\sum p_j - \cancel{\sum z_i}^{\,0}}{n-m} = \frac{-1-2-3}{3} = \boxed{-2}$$

The angles of the asymptotes are given by

$$\alpha = \frac{2\ell+1}{3}\pi, \quad \ell = 0, 1, 2$$

$\underline{\ell=0} \Rightarrow \alpha = \frac{\pi}{3} = 60^\circ$

$\underline{\ell=1} \Rightarrow \alpha = \pi = 180^\circ$

$\underline{\ell=2} \Rightarrow \alpha = \frac{5\pi}{3} = 300^\circ$

The intervals $(-\infty, -3]$ and $[-2, -1]$ are the real part of the RL since they are to the left of an odd number of poles.

The breakaway point is computed using

$$K(s) = \frac{-d(s)}{n(s)} = -(s^3 + 6s^2 + 11s + 6)$$

$$\Rightarrow \frac{dK(s)}{ds} = -3s^2 - 12s - 11 = 0.$$

$$\Rightarrow s = -2 \pm \frac{\sqrt{3}}{3}$$

Choose: $s = -2 + \frac{\sqrt{3}}{3}$ since $-2 + \frac{\sqrt{3}}{3} \in (-2, -1)$

and $-2 - \frac{\sqrt{3}}{3} \notin (-2, -1)$.

Use Routh-Hurwitz table to find crossover points:

$$s^3 + 6s^2 + 11s + (6+K) = 0.$$

$$\begin{array}{c|cc} s^3 & 1 & 11 \\ s^2 & 6 & 6+K \\ s^1 & b_1 = 10 - \frac{K}{6} & 0 \\ s^0 & 6+K & 0 \end{array}$$

$$b_1 = -\frac{1}{6}\begin{vmatrix} 1 & 11 \\ 6 & 6+K \end{vmatrix}$$

$$= -\frac{1}{6}(6+K-66)$$

$$= 10 - \frac{K}{6}$$

we need $10 - \frac{K}{6} > 0$ and $6 + K > 0$

$\Rightarrow K < 60$ and $K > -6$

only need $K < 60$ since $0 < K < \infty$.

So our crossover point is $\boxed{K_c = 60}$